경제학석사 학위논문

A More Efficient Method to Elicit Economic Preferences than the Multiple Price List Design

경제적선호를 추론하는데 있어 다중가격리스트보다 더 나은 방법

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Abstract

Since Holt and Laury (2002) suggested the Multiple Price List design (MPL), MPL has been widely used for economic preference elicitation experiment. But, as Andersen et al. (2006) pointed out, MPL only elicits interval estimates and could be susceptible to framing effect. Even though Harrison et al. (2005) suggested the Iterative Multiple Price List design (iMPL) as a solution for interval estimates, iMPL seems to make an experiment burdensome requiring a subject to make more economic decisions than MPL. In this paper, I suggest a New Method (NM) which can elicit smaller interval estimates than MPL with same or even smaller number of questions. This is possible because NM uses information which is already known and knewly emerged during the experiment. This process is contained in NM in the form of bayesian updating and sequential questioning structure.

Keyword: MPL, iMPL, Economic preference, Experiment, Interval estimate, Sequential questioning structure

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1 Introduction

Since Holt and Laury (2002) (HL) suggested the Multiple Price List design (MPL), MPL has been widely used for economic preference elicitation experiment. But, the popularity of MPL does not guarantee its methodological perfectness. As Andersen et al. (2006) (AN) pointed out, it has some disadvantages. The main disadvantages are that MPL only elicits interval estimates and could be susceptible to framing effects.

For a solution to the interval estimates, AN introduced the Iterative Multiple Price List design (iMPL) which was devised at Harrison et al. (2005) (HA). To elicit smaller length of interval estimates, iMPL asks similar form of MPL questions two times. Thus, subjects are required to make much more economic decisions than MPL.

Asking more decisions for smaller interval estimates seems quite reasonable. But what if it is possible to elicit smaller interval estimates by requiring same or even less number of decisions than MPL? Is there such an unrealistic experimental design? The answer is yes. In this paper, I suggest a new method (NM) that satisfies the very feature.

The main idea of NM is very similar to the optimal behavior of the questioners of "Twenty Questions". In this game, questioners can ask at most finite number of questions, so they carefully choose what to ask. Maybe, they would use all the information they can attain, for example answerer's age, sex, appearance, wedding ring, perfume brand etc. After each question, they update their information and use all the information again to make the next question.

Suppose now that the experimenters are the questioners and the subject is the answerer. Like questioners, the experimenters need to choose what to ask using the attainable information about the economic preferences. After each question, as questioners do, they need to update their information and use all the information again to make the next question.

As I illustrated, the main idea of NM is the usage of the information which is already known and newly emerged during the experiment. This process is contained in NM in the form of bayesian updating and sequential questioning structure.

From the basic concepts of the experiment, I will explain the new method specifically. It might be quite tedious work to follow the series of descriptions. But I believe that it is worthwhile for them who desire to elicit more precise economic preference estimates and do not want to require burdensome tasks to subjects.

In the next section, I will explain the basic concepts by showing the HL's risk preference elicitation experiment as an example. In the third section, the structure of the conventional methods, MPL and iMPL, will be introduced. After that, I will explain the structure and procedure of NM specifically. In the fifth section, an experimental application will be illustrated. lastly, In the sixth, I conclude this paper.

2 Setup

In this section, I set up the basic experimental concepts by showing the HL's risk preference elicitation experiment as an example.

HL conducted an experiment to elicit subject's risk preference. To achieve their goal, they required a subject to make economic decisions. The followings were some of the binary economic decision problems which they presented to a subject.[1]

$\mathbf{Q_1}$. 0.2 \$3.85 0.8 \$0.10 *vs.* 0.2 \$2.00 0.8 \$1.60

$\mathbf{Q_2}$. 0.4 \$3.85 0.6 \$0.10 *vs.* 0.4 \$2.00 0.6 \$1.60

$\mathbf{Q_3}$. 0.6 \$3.85 0.4 \$0.10 *vs.* 0.6 \$2.00 0.4 \$1.60

$\mathbf{Q_4}$. 0.8 \$3.85 0.2 \$0.10 *vs.* 0.8 \$2.00 0.2 \$1.60

$\mathbf{Q_5}$. 0.9 \$3.85 0.1 \$0.10 *vs.* 0.9 \$2.00 0.1 \$1.60

At a glance, these look just simple economic decision problems. But, HL could get information about subject's risk preference from the answers of these. How could they do that? It was possible because these problems were not just simple economic decision problems. Rather, these were carefully designed choice problems under economic theory.

When constructing these problems, HL assumed that subjects are Expected ustility maximizer and their utility functions are a CRRA utility function, $u(x) = \frac{x^{1-\theta}}{1-\theta}$.[2] Under the assumed economic model, the above economic decision problems can be interpreted as follows.[3]

[1]The numbers before the monetary values are probabilities.

[2]Under this utility form, the curvature parameter θ is regarded as a measure for risk preference.

[3]Note that θ is used to denote the true value of subject's risk preference. In this paper, I

$$\mathbf{Q_1}.\ \theta \geq -0.95 \quad vs. \quad \theta \leq -0.95$$

$$\mathbf{Q_2}.\ \theta \geq -0.15 \quad vs. \quad \theta \leq -0.15$$

$$\mathbf{Q_3}.\ \theta \geq 0.41 \quad vs. \quad \theta \leq 0.41$$

$$\mathbf{Q_4}.\ \theta \geq 0.97 \quad vs. \quad \theta \leq 0.97$$

$$\mathbf{Q_5}.\ \theta \geq 1.37 \quad vs. \quad \theta \leq 1.37.$$

This interpretation of an economic decision problem can be regarded as an underlying question and every economic decision problem corresponds to an underlying question under assumed economic theory. HL made this underlying questions first, and then converted these to specific economic decision problems. Since an economic decision problem is just a converted form of an underlying question, from now on, I only consider the underlying questions. The concept of underlying questions take a crucial role in this paper, so I formally define this concept.

Definition 1. (A question, Q**)**

Real value Q denotes a question that asks the following binary choice to a subject.

$$\theta \leq Q \quad \text{or} \quad \theta \geq Q$$

The points of questions above, $\{-0.95, -0.15, 0.41, 0.97, 1.37\}$, were selected by HL. They set these values to divide the searching area which is the

will use θ as a notation for true value of an economic preference. In addition to the notation θ, I will use T^θ as a notation for an interval that contains θ and let T^θ be called a true interval.

interval from the smallest point to the largest point, $[-0.95, 1.37]$, equally as possible. By setting the points in this fashion, they wanted to search θ through the entire searching area with equal weight. The selected questions form a set of questions. Let this set be denoted by $\mathbb{Q}$ and the smallest and the largest Q in $\mathbb{Q}$ be the lower and the upper boundary questions and denoted by $\underline{Q}$ and $\overline{Q}$, respectively. $\mathbb{Q}$ of HL's questions can be represented as follows.

$$\mathbb{Q}^{HL} = \{-0.95, -0.15, 0.41, 0.97, 1.37\}$$

HL also decided how to present their questions to a subject. In their experiment, they decided to present the questions all at once. In fact, there are a lot of ways to present questions to a subject. For example, $\mathbb{Q}$ can be presented all at one stage like HL's case or sequentially for several stages.[4] Each way forms a structure and this can be represented by a matrix. This concept of a structure is worth enough to be a definition.

Definition 2. (A structure of $\mathbb{Q}$, $\mathbb{S}$)

A structure of $\mathbb{Q}$, denoted by $\mathbb{S}$, is a set of $I \times J$ matrices where I and J are the maximum number of stages and questions at a stage. Each row i and column j element of a matrix in $\mathbb{S}$ represents the value Q of j-th question at i-th stage. If there is no question for that i, j case, the value of element would be imaginary number i. Imaginary number i is used here to avoid confusion with real values of Q.

[4]Here, I used a concept of a stage. A stage is a natural number and one stage consists of experimenter's action and consecutive subject's reaction. For example, experimenter's asking(action) and consecutive subject's answering(reaction) consist of one stage. Let the number of stages be denoted by $\mathbf{S}$.

The structure of $\mathbb{Q}^{HL}$ can be represented as follows.

$$\mathbb{S}^{HL} = \left\{ \begin{pmatrix} -0.95 & -0.15 & 0.41 & 0.97 & 1.37 \end{pmatrix} \right\}$$

From the subject's answer's of the economic decision problems, HL could deduce T^{θ}. For example, if a subject's answers were (0,1,1,1,1)[5], then they could deduce that the subject's T^{θ} is $[-0.95, -0.15]$. This HL's experimental setting so far is called the Multiple Price List design (MPL).

As a result of MPL, HL got interval estimates of subjects' risk preference, not point estimates. They estimated the point estimates using the maximum likelihood estimation. When doing estimation, broader length of interval estimates is related to more risk of getting wrong point estimates.[6] AN pointed out this shortcoming and suggested iMPL which was devised by HA as a solution.

To elicit smaller length of interval estimates, iMPL asks similar form of MPL questions two times. For example, if a subject's answers of the above 5 questions were (0,1,1,1,1), then T^{θ} from the answers is $[-0.95, -0.15]$, which is same as MPL's interval estimate above. Using this information about T^{θ}, iMPL asks same form of MPL questions which has values in $[-0.95, -0.15]$ and divide this interval equally as possible, $\{-0.82, -0.68, -0.55, -0.42, -0.28\}$. Suppose that subject's answers from the second stage's questions were (0,0,0,1,1), then T^{θ} from this answer is $[-0.55, -0.42]$ which has much smaller length than MPL's interval estimate, $[-0.95, -0.15]$.

Even though iMPL can elicit shorter interval estimates than MPL, this makes experiment burdensome by requiring more economic decision to a subject. Thus, there is a trade-off between the length of interval estimate and the

[5] 0 and 1 mean the formal and latter option, respectively.

[6] This relation is described at Appendix A.

burdensomeness of the experiment. But what if the incompatible features of the trade-off became compatible ones? It would be really good news to both the experimenters and subjects. In this paper, I want to show you that the incompatibility can be broken by a new experimental design. This new design makes it possible to get shorter length of interval estimate requiring same or even less economic decisions than MPL.

To explain the New Method (NM), few more concepts are needed. First of all, I introduce a concept of a history.

Definition 3. (A history, h)

A history, denoted by h, is a J-tuple where J is the number of previous stages. Each j-th leftmost element could be translated to the record of answers at the j-th stage by representing the value of j-th leftmost element as a binary number. Each i-th leftmost element of the translated binary number indicates the answer from i-th question at that stage. If there is no question in some stages, the values of elements for those stages are the imaginary number i. If every stage consists of only one question, each element of h will be 0 or 1. In this case, J elements of h can be regarded as a binary number and represented as corresponding decimal number. For concise notation, Let h which is 1-tuple be represented without parentheses.

The subject's answers from the above iMPL case can be represented as follows.

$$\text{A history of a subject's answer,}\ \ h \ = \ (15, 3)$$

Record of subject's answers at 1-th stage

$$:\ 15 \ = \ 2^4 \times 0 + 2^3 \times 1 + 2^2 \times 1 + 2^1 \times 1 + 2^0 \times 1 \ = \ 01111_{(2)}$$

Record of subject's answers at 2-th stage

$$: \; 3 \;=\; 2^4 \times 0 + 2^3 \times 0 + 2^2 \times 0 + 2^1 \times 1 + 2^0 \times 1 \;=\; 00011_{(2)}$$

The last four line of the definition of a history is made for the purpose of concise notation of NM. Hoping for you to get familiar with the notation easily, I present an example of the last four line.

Example (A history, 1 question at each stage, $\mathbf{S} = 5$)

A history of a subject's answer, $h \;=\; (0, 1, 1, 1, 0)$

Record of subject's answers at 1-th stage $: \; 0 \;=\; 2^0 \times 0 \;=\; 0_{(2)}$

Record of subject's answers at 2-th stage $: \; 1 \;=\; 2^0 \times 1 \;=\; 1_{(2)}$

Record of subject's answers at 3-th stage $: \; 1 \;=\; 2^0 \times 1 \;=\; 1_{(2)}$

Record of subject's answers at 4-th stage $: \; 1 \;=\; 2^0 \times 1 \;=\; 1_{(2)}$

Record of subject's answers at 5-th stage $: \; 0 \;=\; 2^0 \times 0 \;=\; 0_{(2)}$

$$h \;=\; (0, 1, 1, 1, 0) \;=\; (01110_{(2)}) \;=\; (14) \;=\; 14$$

It is also necessary to clarify the difference between two similar concepts, the round and stage. Firstly, the number of round, denoted by **R**, is a natural number and one round consists of three parts which are experimenter's action, consecutive subject's reaction and experimenter's reaction against the subject's reaction. **R** can be turned to the next number when experimenter's reaction depends on subject's previous answers. If not, **R** will never be turned to the next number even though there are several experimenter's reactions. But the stage, **S**, only counts the number of pairs of experimenter's action and subject's reaction.

The MPL's questions were presented all at once to a subject, thus the first experimenter's action was asking all the questions. The consecutive subject's reaction was answering and the experimenter's reaction against the subject's reaction was finishing the questioning. Here, experimenter finished the questioning regardless of previous subject's answers, thus the last experimenter's reaction didn't depend on subject's reaction. And there was only one pair of experimenter's action and subject's reaction. So, MPL has $\mathbf{R} = 1$ and $\mathbf{S} = 1$.

In fact, the above MPL's 5 questions could have been presented for several stages, one question for each stage. In this case, $\mathbf{S} = 5$. But, note that $\mathbf{R}$ will never be turned to the next number because each experimenter's consecutive question against subject's answer does not depend on previous subject's answers. Thus $\mathbf{R} = 1$ in MPL's case, even there are several actions and reactions.

Since we've just defined the concept of history and explained the difference between round and stage, we can represent more abundant situation by using our notations. Let the subscript h and superscript s, o of $Q_h^{s,o}$ denote history, the stage number and order of the question at that stage, respectively. This notation and concept of the round make it possible to represent structure which has more than one round. I present an example.

Example ($\mathbb{Q}$ and $\mathbb{S}$ with 1 question at each stage, $\mathbf{R} = 3$ and $\mathbf{S} = 3$)

$$\begin{aligned}
\mathbb{Q} &= \left\{ Q^{1,1}, Q_0^{2,1}, Q_1^{2,1}, Q_0^{3,1}, Q_1^{3,1}\, Q_2^{3,1}, Q_3^{3,1} \right\} \\
\mathbb{S} &= \left\{ \begin{pmatrix} Q^{1,1} \\ Q_0^{2,1} \\ Q_0^{3,1} \end{pmatrix}, \begin{pmatrix} Q^{1,1} \\ Q_0^{2,1} \\ Q_1^{3,1} \end{pmatrix}, \begin{pmatrix} Q^{1,1} \\ Q_1^{2,1} \\ Q_2^{3,1} \end{pmatrix}, \begin{pmatrix} Q^{1,1} \\ Q_1^{2,1} \\ Q_3^{3,1} \end{pmatrix} \right\}
\end{aligned}$$

The next concept needed is about a priori knowledge. As we saw in the HL's design, they set the searching area to be $[-0.95, 1.37]$. Why did they set this particular interval for the searching area? was that chosen arbitrary? Or did they used some information about θ? In fact, they used risk preference values reported in previous econometric analysis of auction data as a reference for choosing the searching area.

As HL did, a priori knowledge can be used to set the bounded searching area. Furthermore, this bounded searching area can be converted to a form of prior distribution of θ, $\pi(\theta)$. In HL's case, the searching area can be converted to the uniform distribution, $\pi(\theta) \sim \text{uniform}[-0.95, 1.37]$.

Suppose that HL could have attained the experimental data from similar previous experiment. Then $\pi(\theta)$ could have been updated to posterior distribution of θ, $\pi(\theta|\mathrm{X})$, using a proper likelihood function.[7]

Using $\pi(\theta|\mathrm{X})$, $\mathbb{Q}$ which is constructed by an experimenter can be evaluated in two perspectives. The first is how many subjects $\mathbb{Q}$ will fail to elicit T^θ which has finite length and the second is how long the expected length of T^θ would be when only T^θs which have finite length are considered.

Regarding the first evaluation of $\mathbb{Q}$, suppose that a subject has θ that is not in $[\underline{Q}, \overline{Q}]$. Answers from this subject would indicate that T^θ has infinite length, $[-\infty, \underline{Q}]$ or $[\overline{Q}, \infty]$. Under $\pi(\theta|\mathrm{X})$, the probability that infinite length of T^θ might occur is $P(\theta < \underline{Q}$ or $\overline{Q} < \theta)$. Because each subject has only one θ as the true value, this probability represents how much proportion of subjects an experimenter will fail to elicit T^θ that has finite length under the constructed $\mathbb{Q}$. This concept looks like a loss to an experimenter, so let this be the loss level and denoted by α.

$$\alpha \quad = \quad 1 - \int_{\underline{Q}}^{\overline{Q}} \pi(t|\mathrm{X})\, dt.$$

[7]From now on, only the posterior distribution will be used for explanation for conciseness.

When constructing $\mathbb{Q}$, an experimenter can adjust α by trimming boundary questions. After setting α, the domain of $\pi(\theta|\mathrm{X})$ can be truncated on $[\underline{Q}, \overline{Q}]$ making the truncated posterior distribution of θ, $t_{[\underline{Q},\overline{Q}]}(\theta|\mathrm{X})$.

$$t_{[\underline{Q},\overline{Q}]}(\theta|\mathrm{X}) \quad = \quad \left[\int_{\underline{Q}}^{\overline{Q}} \pi(t|\mathrm{X})\, dt\right]^{-1} \pi(\theta|\mathrm{X})\,, \ \ \underline{Q} \le \theta \le \overline{Q}$$

Let $t_{\mathrm{T}^\theta}(\theta|\mathrm{X})$ also denotes posterior distribution which is truncated on T^θ. Capital italic letter, T, will be used to denote CDF form.

The second evaluation of $\mathbb{Q}$ is about how long the expected length of T^θ would be when only T^θs which have finite length are considered. This concept is quite important, I formally define this.

Definition 4. (The expected length of $\mathbf{T}^\theta$ of $\mathbb{Q}$, ELTI($\mathbb{Q}$))

The expected length of T^θ of $\mathbb{Q}$ is denoted by ELTI($\mathbb{Q}$) and defined as[8]

$$\begin{aligned} \mathrm{ELTI}(\mathbb{Q}) \quad &= \quad \sum_{i=1}^{|\mathbb{Q}|-1} \left[T_{[\underline{Q},\overline{Q}]}(Q_{i+1}|\mathrm{X}) - T_{[\underline{Q},\overline{Q}]}(Q_i|\mathrm{X})\right](Q_{i+1} - Q_i) \\ Q_i \ &\in \ \mathbb{Q} \ \ for \ \ i = 1, \cdots, |\mathbb{Q}| \\ Q_{i+1} \ &> \ Q_i \ \ for \ \ i = 1, \cdots, |\mathbb{Q}| - 1 \end{aligned}$$

As I mentioned earlier, when doing estimation, broader length of interval estimate is related to the more risk of getting wrong point estimate. With the same logic, broader expected length of interval estimate is related to more expected risk of getting wrong point estimate.[9]

In terms of those two evaluation of $\mathbb{Q}$ and the burdensomeness of an experiment, efficiency and betternees of $\mathbb{Q}$ can be thought. For arbitrary two

[8]Note that the brackets are used to separate probability caculation. I will repeat the same thing in the later sections.

[9]This relation is also described at appendix A.

sets of questions $\mathbb{Q}$ and $\mathbb{Q}'$ with same boundary questions, $\mathbb{Q}$ is more efficient than $\mathbb{Q}'$ if and only if ELTI($\mathbb{Q}$) is smaller than or equal to ELTI($\mathbb{Q}'$) when the number of questions of $\mathbb{Q}$ which are asked to a subject during the experiment is less than or equal to that of $\mathbb{Q}'$. Also $\mathbb{Q}$ is better than $\mathbb{Q}'$ if and only if ELTI($\mathbb{Q}$) is smaller than ELTI($\mathbb{Q}'$) when the number of questions of $\mathbb{Q}$ is larger than that of $\mathbb{Q}'$.

According to these concepts, $\mathbb{Q}^{iMPL}$ is better but not more efficient than $\mathbb{Q}^{MPL}$. In this paper, I want to suggest a New Method (NM) that can generate $\mathbb{Q}^{NM}$ which is more efficient than $\mathbb{Q}^{MPL}$.

3 Conventional Methods: MPL and iMPL

In this section, the structures of conventional methods, MPL and iMPL, will be described.[10]

3.1 The Multiple Price List Design, MPL

MPL asks all the questions at 1-th stage and deduce T^{θ} from the answers. $\mathbb{Q}$, $\mathbb{S}$, ELTI($\mathbb{Q}$) of MPL with N questions are as follows,

$$\begin{aligned}
\mathbb{Q}^{MPL} &= \{\, Q^{1,i} \mid Q^{1,i} \in \mathbb{R} \;\; for \;\; i = 1, 2, \cdots, N \\
&\qquad\qquad s.t. \;\; Q^{1,j+1} - Q^{1,j} = \frac{Q^{1,N} - Q^{1,1}}{N-1} \\
&\qquad\qquad for \;\; j = 1, 2, \cdots, N-1 \,\}
\end{aligned}$$

$$\mathbb{S}^{MPL} = \left\{ \begin{pmatrix} Q^{1,1} & Q^{1,2} & \ldots & Q^{1,N-1} & Q^{1,N} \end{pmatrix} \right\}$$

$$\begin{aligned}
\mathrm{ELTI}(\mathbb{Q}^{MPL}) &= \sum_{i=1}^{N-1} \left[T_{[\underline{Q},\overline{Q}]}(Q^{1,i+1}) - T_{[\underline{Q},\overline{Q}]}(Q^{1,i}) \right] \left(Q^{1,i+1} - Q^{1,i} \right) \\
&= \left(\frac{\overline{Q} - \underline{Q}}{N-1} \right) \sum_{i=1}^{N-1} \left[T_{[\underline{Q},\overline{Q}]}(Q^{1,i+1}) - T_{[\underline{Q},\overline{Q}]}(Q^{1,i}) \right] \\
&= \frac{\overline{Q} - \underline{Q}}{N-1}
\end{aligned}$$

In fact, the questions of MPL can be asked sequentially for several stages. Then **S** would be N. But **R** of MPL would never be turned to the next number, because experimenter's reaction does not depend on subject's answer. This implies that MPL does not use newly emerged subject's information during the experiment.

[10] I only consider the case that don't have multiple switching.

3.2 The Iterative Multiple Price List Design, iMPL

iMPL asks questions sequentially for two stages. The second stage's questions are determined according to the first stage's answers. So **R** of iMPL is 2. $\mathbb{Q}$, $\mathbb{S}$, ELTI($\mathbb{Q}$) of iMPL with N_1 questions at 1-th stage and N_2 questions at 2-th stage are as follows.

$$\begin{aligned}
\mathbb{Q}^{iMPL} \;=\; \{\, & Q^{1,i}, Q^{2,j}_{\sum_{k=1}^{i} 2^{N_1+1-k}} \mid Q^{1,i}, Q^{2,j}_{\sum_{k=1}^{i} 2^{N_1+1-k}} \in \mathbb{R} \\
& for \;\; i = 1,2,\cdots,N_1, \;\; j = 1,2,\cdots,N_2 \\
& s.t. \;\; Q^{1,k+1} - Q^{1,k} = \frac{Q^{1,N_1} - Q^{1,1}}{N_1 - 1}, \\
& \quad Q^{2,l+1}_{\sum_{m=1}^{k} 2^{N_1+1-m}} - Q^{2,l}_{\sum_{m=1}^{k} 2^{N_1+1-m}} = \frac{Q^{1,k+1} - Q^{1,k}}{N_2 + 1}, \\
& \quad Q^{2,0}_{\sum_{m=1}^{k} 2^{N_1+1-m}} = Q^{1,k}, \;\; Q^{2,N_2+1}_{\sum_{m=1}^{k} 2^{N_1+1-m}} = Q^{1,k+1} \\
& \quad for \;\; k = 1,2,\cdots, \;\; N_1 - 1, \;\; l = 0,1,2,\cdots,N_2 \,\}
\end{aligned}$$

$$\begin{aligned}
\mathbb{S}^{iMPL} \;=\; & \left\{ \begin{pmatrix} Q^{1,1} & \cdots & Q^{1,N_1} \\ Q^{2,1}_{2^{N_1}} & \cdots & Q^{2,N_2}_{2^{N_1}} \end{pmatrix}, \begin{pmatrix} Q^{1,1} & \cdots & Q^{1,N_1} \\ Q^{2,1}_{2^{N_1}+2^{N_1-1}} & \cdots & Q^{2,N_2}_{2^{N_1}+2^{N_1-1}} \end{pmatrix}, \;\cdots, \right. \\
& \left. \begin{pmatrix} Q^{1,1} & \cdots & Q^{1,N_1} \\ Q^{2,1}_{\sum_{m=1}^{N_1} 2^{N_1+1-m}} & \cdots & Q^{2,N_2}_{\sum_{m=1}^{N_1} 2^{N_1+1-m}} \end{pmatrix} \right\}
\end{aligned}$$

ELTI($\mathbb{Q}^{iMPL}$)

$$\begin{aligned}
= \; & \sum_{i=1}^{N_1-1} \left[T_{[\underline{Q},\overline{Q}]}(Q^{1,i+1}) - T_{[\underline{Q},\overline{Q}]}(Q^{1,i}) \right] \\
& \left(\sum_{j=0}^{N_2} \left[T_{[Q^{1,i+1},Q^{1,i}]}(Q^{2,j+1}_{\sum_{k=1}^{i} 2^{N_1+1-k}}) - T_{[Q^{1,i+1},Q^{1,i}]}(Q^{2,j}_{\sum_{k=1}^{i} 2^{N_1+1-k}}) \right] \right. \\
& \left. \left(Q^{2,j+1}_{\sum_{k=1}^{i} 2^{N_1+1-k}} - Q^{2,j}_{\sum_{k=1}^{i} 2^{N_1+1-k}} \right) \right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{N_1-1} \sum_{j=0}^{N_2} \left[T_{[\underline{Q},\overline{Q}]}(Q^{2,j+1}_{\sum_{k=1}^{i} 2^{N_1+1-k}}) - T_{[\underline{Q},\overline{Q}]}(Q^{2,j}_{\sum_{k=1}^{i} 2^{N_1+1-k}}) \right] \\
&\qquad \left(Q^{2,j+1}_{\sum_{k=1}^{i} 2^{N_1+1-k}} - Q^{2,j}_{\sum_{k=1}^{i} 2^{N_1+1-k}} \right) \\
&= \sum_{i=1}^{N_1-1} \left(\frac{Q^{1,i+1} - Q^{1,i}}{N_2+1} \right) \\
&\qquad \left(\sum_{j=0}^{N_2} \left[T_{[\underline{Q},\overline{Q}]}(Q^{2,j+1}_{\sum_{k=1}^{i} 2^{N_1+1-k}}) - T_{[\underline{Q},\overline{Q}]}(Q^{2,j}_{\sum_{k=1}^{i} 2^{N_1+1-k}}) \right] \right) \\
&= \left(\frac{Q^{1,N_1} - Q^{1,1}}{(N_2+1)(N_1-1)} \right) \\
&\qquad \left(\sum_{i=1}^{N_1-1} \sum_{j=0}^{N_2} \left[T_{[\underline{Q},\overline{Q}]}(Q^{2,j+1}_{\sum_{k=1}^{i} 2^{N_1+1-k}}) - T_{[\underline{Q},\overline{Q}]}(Q^{2,j}_{\sum_{k=1}^{i} 2^{N_1+1-k}}) \right] \right) \\
&= \frac{Q^{1,N_1} - Q^{1,1}}{(N_2+1)(N_1-1)}
\end{aligned}
$$

Since MPL and iMPL divide $[\underline{Q},\overline{Q}]$ to equal length of intervals, their ELTI is independent of $\pi(\theta)$ or $\pi(\theta|\mathrm{X})$. This independency of the conventional methods have the pros and cons. The strength of the independency is situational robustness of the measure, that is, regardless of the distribution of θ this measure always produces fixed length of T^{θ}. However, the independency can also be regarded as wasting information which can be useful to make a more efficient measure.

In the next section, a new method that uses such information will be introduced. This use of the information allow an experimenter to construct more efficient $\mathbb{Q}$.

4 A More Efficient Method

In this section, I introduce a new method (NM). $\mathbb{Q}$ constructed by NM will elicit smaller ELTI than $\mathbb{Q}^{MPL}$ with same or smaller number of questions which are asked to a subject.

The followings are the procedures of NM.

① Construction of prior and posterior distribution

② Selection of the loss level α and the maximum number of questions to be asked

③ Construction of $\mathbb{Q}^{NM}$

- Using the attainable information (NM_1)
- Using the information that is attainable and emerged during experiment (NM_2)

④ Conversion of $\mathbb{Q}^{NM}$ to specific binary economic decision problems

I will explain each procedure serially.

4.1 Construction of prior and posterior distribution

In contrast to MPL, prior distribution $\pi(\theta)$ or posterior distribution $\pi(\theta|\mathrm{X})$ take an important role in NM. So attainable information need to be converted to $\pi(\theta)$ and data from previous similar experiments is required to be updated to construct $\pi(\theta|\mathrm{X})$. Specific procedure will be demonstrated in section 5.

4.2 Selection of the loss level α and the maximum number of questions to be asked

After construction of $\pi(\theta)$ or $\pi(\theta|\mathrm{X})$, an experimenter should set the loss level α and the maximum number of questions to be asked.

4.3 Construction of $\mathbb{Q}^{NM}$

In NM, α acts like a criterion for selecting boundary questions. $\underline{Q}$ and $\overline{Q}$ should be selected to satisfy the following conditions.

$$\begin{aligned}\alpha &\geq 1 - \int_{\underline{Q}}^{\overline{Q}} \pi(\theta|\mathrm{X})\,dx \\ \int_{\underline{Q}}^{\overline{Q}} \theta\, t_{[\underline{Q},\overline{Q}]}(\theta|\mathrm{X})\,d\theta &= \int_{-\infty}^{\infty} \theta\,\pi(\theta|\mathrm{X})\,d\theta\end{aligned}$$

The second equation guarantees unbiasedness of $\mathrm{E}[\theta]$ under the truncated distribution $t_{[\underline{Q},\overline{Q}]}(\theta)$.

What's left to construct $\mathbb{Q}^{NM}$ is to select specific values of questions which are not boundary questions. Let these questions be called inner questions. NM has two forms of structure for being applied to various situations. The first structure (NM_1) presents all the questions at one stage like MPL and the second structure (NM_2) has several rounds and presents only one question at each stage.

4.3.1 Using the attainable information (NM_1)

The $\mathbb{Q}$ and $\mathbb{S}$ of NM_1 with N questions are as follows

$$\mathbb{Q}^{NM\_1} = \{Q^{1,1}, Q^{1,2}, \cdots, Q^{1,N}\}$$

$$\mathbb{S}^{NM\_1} = \left\{\begin{pmatrix} Q^{1,1} & Q^{1,2} & \cdots & Q^{1,N}\end{pmatrix}\right\}$$

Inner questions in $\mathbb{Q}^{NM\_1}$ are selected to minimize ELTI as possible. These values can be found by solving the following equation.

$$\mathbb{Q}^* = \mathop{\arg\min}_{\substack{\mathbb{Q}\in[\underline{Q},\overline{Q}]^N \\ Q^{1,1}=\underline{Q} \\ Q^{1,N}=\overline{Q}}} \mathrm{ELTI}(\mathbb{Q}).$$

This is $N-2$-dimensional optimization problem and existence and uniqueness of the $\mathbb{Q}^*$ depends on $\pi(\theta|\mathrm{X})$. Since finding the global optimum values is quite complicate problem, I approached this problem from another direction. Starting from $\mathbb{Q}$ which divides $[\underline{Q}, \overline{Q}]$ equally, I applied an improvement algorithm to $\mathbb{Q}$ iteratively to get $\mathbb{Q}^*$ that has smaller ELTI than that of $\mathbb{Q}$. I introduce this improvement algorithm.

The Improvement Algorithm (IA)

There are many optimization algorithms such as the Newton's method and quasi Newton's methods. But these methods require the objective function to be differentiable. Although the algorithm I introduce here can be applied to only particular case, this do not require differentiability of the objective function. So this algorithm could be applied to not only differentiable objective functions but also ones which are not differentiable.

IA consists of two sub-algorithms, the first sub-algorithm (IA_1) is a new one that finds global optimum in 1-dimensional sub-problem and the second sub-algorithm is the coordinate descent algorithm (IA_2). Since IA_2 is just repeated application of IA_1, IA_1 is the core of IA.

The first sub-algorithm (IA_1)

IA_1 finds global optimum value numerically within the predetermined tolerance in 1-dimensional sub-problem. From now on, let $\mathbb{Q}$ have just three questions, two boundary questions which are already selected by α and one inner question.

$$\begin{aligned}\mathbb{Q} &= \{Q^{1,1},\ Q^{1,2},\ Q^{1,3}\},\ \ Q^{1,1} = \underline{Q} \text{ and } Q^{1,3} = \overline{Q} \\ &= \{\underline{Q},\ Q,\ \overline{Q}\}\end{aligned}$$

In this case, finding the optimal $\mathbb{Q}^*$ is same as finding the optimal inner question Q^*. Thus let the notation $\mathrm{ELTI}(Q)$ be same as $\mathrm{ELTI}(\mathbb{Q})$.

$$
\begin{aligned}
\mathbb{Q}^* \quad &= \quad \mathop{\arg\min}_{\substack{\mathbb{Q} \in [\underline{Q}, \overline{Q}]^3 \\ Q^{1,1} = \underline{Q} \\ Q^{1,3} = \overline{Q}}} \mathrm{ELTI}(\mathbb{Q}) \\
&= \quad \mathop{\arg\min}_{\substack{\mathbb{Q} \in [\underline{Q}, \overline{Q}]^3 \\ Q^{1,1} = \underline{Q} \\ Q^{1,3} = \overline{Q}}} T_{[\underline{Q},\overline{Q}]}(Q)\,(Q - \underline{Q}) \;+\; \left[1 - T_{[\underline{Q},\overline{Q}]}(Q)\right]\,(\overline{Q} - Q) \\
&\Leftrightarrow \quad \mathop{\arg\min}_{Q \in [\underline{Q}, \overline{Q}]} T_{[\underline{Q},\overline{Q}]}(Q)\,(Q - \underline{Q}) \;+\; \left[1 - T_{[\underline{Q},\overline{Q}]}(Q)\right]\,(\overline{Q} - Q) \\
&= \quad \mathop{\arg\min}_{Q \in [\underline{Q}, \overline{Q}]} \mathrm{ELTI(Q)} \quad = \quad Q^*
\end{aligned}
$$

$\mathrm{ELTI}(Q)$ can be represented as a concise way and this representation makes comparison of $\mathrm{ELTI}(Q)$ quite simple.

$\mathrm{ELTI}(Q)$

$$
\begin{aligned}
&= \; T_{[\underline{Q},\overline{Q}]}(Q)\,(Q - \underline{Q}) \;+\; \left[1 - T_{[\underline{Q},\overline{Q}]}\right]\,(\overline{Q} - Q) \\
&= \; \left[T_{[\underline{Q},\overline{Q}]}(Q) - \frac{1}{2}\right](Q - \underline{Q}) + \frac{1}{2}(Q - \underline{Q}) \\
&\qquad\qquad + \left[\frac{1}{2} - T_{[\underline{Q},\overline{Q}]}\right](\overline{Q} - Q) + \frac{1}{2}(\overline{Q} - Q) \\
&= \; \left[T_{[\underline{Q},\overline{Q}]}(Q) - \frac{1}{2}\right](Q - \underline{Q}) \\
&\qquad\qquad + \left[\frac{1}{2} - T_{[\underline{Q},\overline{Q}]}\right](\overline{Q} - Q) + \frac{1}{2}(\overline{Q} - \underline{Q}) \\
&= \; \left[T_{[\underline{Q},\overline{Q}]}(Q) - \frac{1}{2}\right]\left(Q - \frac{\overline{Q} + \underline{Q}}{2}\right) + \left[T_{[\underline{Q},\overline{Q}]}(Q) - \frac{1}{2}\right]\left(\frac{\overline{Q} + \underline{Q}}{2} - \underline{Q}\right) \\
&\quad + \left[\frac{1}{2} - T_{[\underline{Q},\overline{Q}]}(Q)\right]\left(\overline{Q} - \frac{\overline{Q} + \underline{Q}}{2}\right) + \left[\frac{1}{2} - T_{[\underline{Q},\overline{Q}]}(Q)\right]\left(\frac{\overline{Q} + \underline{Q}}{2} - Q\right) \\
&\quad + \frac{1}{2}(\overline{Q} - \underline{Q}) \\
&= \; 2\left[T_{[\underline{Q},\overline{Q}]}(Q) - \frac{1}{2}\right]\left(Q - \frac{\overline{Q} + \underline{Q}}{2}\right) + \frac{1}{2}(\overline{Q} - \underline{Q})
\end{aligned}
$$

Since the second term of the last line is common for arbitrary $\mathbb{Q}$ which has same boundary questions, comparison of ELTIs is reduced to comparison of the first terms. Thus, for arbitrary $\mathbb{Q}$ and $\mathbb{Q}'$ that have same boundary questions, the followings are true.

$$\begin{aligned}
&\mathrm{ELTI}(\mathbb{Q}) \le \mathrm{ELTI}(\mathbb{Q}'),\ \mathbb{Q} = \{\underline{Q},\ Q,\ \overline{Q}\},\ \mathbb{Q}' = \{\underline{Q},\ Q',\ \overline{Q}\} \\
&\Leftrightarrow\ \mathrm{ELTI}(Q) \le \mathrm{ELTI}(Q') \\
&\Leftrightarrow\ \left[T_{[\underline{Q},\overline{Q}]}(Q) - \frac{1}{2}\right]\left(Q - \frac{\overline{Q}+\underline{Q}}{2}\right) \le \left[T_{[\underline{Q},\overline{Q}]}(Q') - \frac{1}{2}\right]\left(Q' - \frac{\overline{Q}+\underline{Q}}{2}\right)
\end{aligned}$$

For concise notation, I define this as the reduced form of ELTI(Q).

Definition 5. (The reduced form of ELTI(Q), RE(Q))

The reduced form of ELTI(Q), denoted by RE(Q), is defined as

$$\mathrm{RE}(Q) = \left[T_{[\underline{Q},\overline{Q}]}(Q) - \frac{1}{2}\right]\left(Q - \frac{\overline{Q}+\underline{Q}}{2}\right)$$

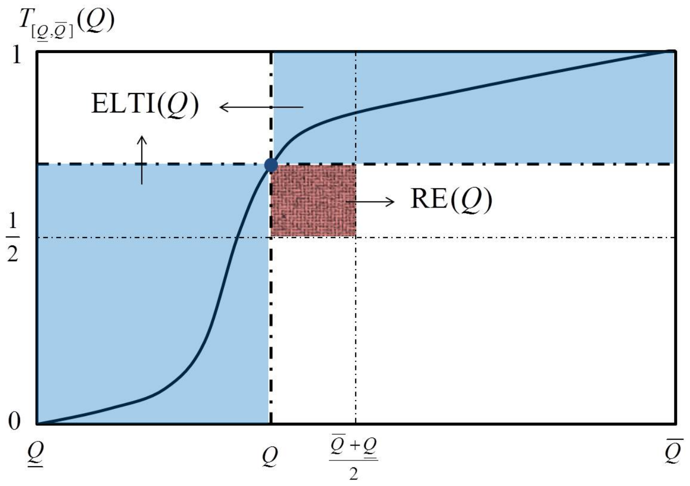


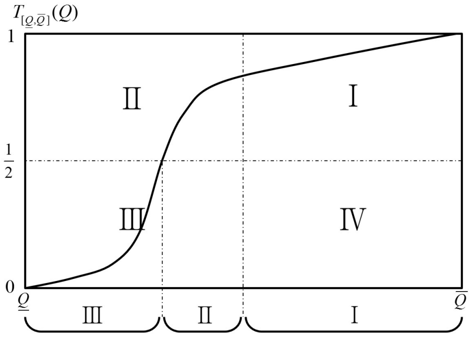
Figure 1: Graphic representation of ELTI(Q) and RE(Q)

To easily explain how IA_1 works, it is good idea to depict ELTI(Q) and RE(Q) on the XY-plane (Figure 1). X-axis and Y-axis represent the value of

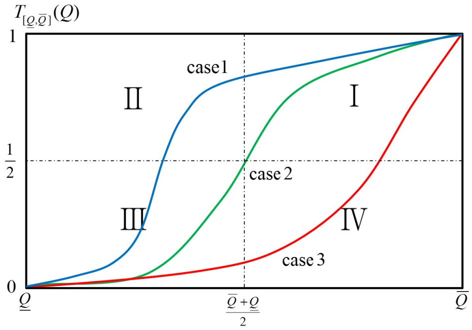
Q and cumulative truncated distribution $T_{[\underline{Q},\overline{Q}]}(Q)$, respectively.

Graphic representation of ELTI(Q) and RE(Q) shows that comparison of RE(Q) would be much simpler than that of ELTI(Q). (Figure 1) Note that RE(Q) can be negative value according to the form of $T_{[\underline{Q},\overline{Q}]}(Q)$ and the value of Q.

IA_1 finds the global optimal question Q^* by eliminating less optimal questions. To eliminate less optimal questions, I present two propositions. For the proofs of the propositions, the area of the graph needs to be partitioned. The whole area can be partitioned to four parts. (Figure 2)



(a) Partitioned area and corresponding questions



(b) Three possible cases of CDF

Figure 2: Partitioned area and corresponding questions, Three possible cases of cdf

$$\begin{aligned}
\mathrm{I} \ &: \ Q \geq \frac{\overline{Q}+\underline{Q}}{2} \quad \textit{and} \quad T_{[\underline{Q},\overline{Q}]}(Q) \geq \frac{1}{2} \\
\mathrm{II} \ &: \ Q < \frac{\overline{Q}+\underline{Q}}{2} \quad \textit{and} \quad T_{[\underline{Q},\overline{Q}]}(Q) \geq \frac{1}{2} \\
\mathrm{III} \ &: \ Q < \frac{\overline{Q}+\underline{Q}}{2} \quad \textit{and} \quad T_{[\underline{Q},\overline{Q}]}(Q) < \frac{1}{2} \\
\mathrm{IV} \ &: \ Q \geq \frac{\overline{Q}+\underline{Q}}{2} \quad \textit{and} \quad T_{[\underline{Q},\overline{Q}]}(Q) < \frac{1}{2}
\end{aligned}$$

Note that any CDF must be one of the following three cases. (Figure 2) The first case of CDF passes through I, II and III regions. The second and third cases pass through I, III and I, III, IV regions, respectively. Since The first and third cases are just reflection of each other, I will explain only the first case. In the second case, the optimal question Q^* can be easily found by the next propositions. In the next propositions, Q_{mid} and Q_{med} denote the middle point and median.

Proposition 1.

$$\begin{aligned}&\forall\ Q\ \in\ \mathrm{I}\ \mathit{or}\ \mathrm{III},\\&\mathrm{RE}(Q)\geq\mathrm{RE}(Q_{mid})\quad and\quad \mathrm{RE}(Q)\geq\mathrm{RE}(Q_{med})\end{aligned}$$

Proof.

$$\begin{aligned}\mathrm{RE}(Q_{mid}) &= \left[T_{[\underline{Q},\overline{Q}]}(Q_{mid})-\frac{1}{2}\right]\left(Q_{mid}-\frac{\overline{Q}+\underline{Q}}{2}\right) = 0\\ \mathrm{RE}(Q_{med}) &= \left[T_{[\underline{Q},\overline{Q}]}(Q_{med})-\frac{1}{2}\right]\left(Q_{med}-\frac{\overline{Q}+\underline{Q}}{2}\right) = 0\\ \forall\ Q\ &\in\ \mathrm{I}\ \mathit{or}\ \mathrm{III},\\ \mathrm{RE}(Q) &= \left[T_{[\underline{Q},\overline{Q}]}(Q)-\frac{1}{2}\right]\left(Q-\frac{\overline{Q}+\underline{Q}}{2}\right) \geq 0\end{aligned}$$

□

Proposition 2.

$$\begin{aligned}&\forall\ Q\ \in\ \mathrm{II}\ \mathit{or}\ \mathrm{IV},\\&\mathrm{RE}(Q)\leq\mathrm{RE}(Q_{mid})\quad and\quad \mathrm{RE}(Q)\leq\mathrm{RE}(Q_{med})\end{aligned}$$

Proof.

$$\begin{aligned}\mathrm{RE}(Q_{mid}) &= \left[T_{[\underline{Q},\overline{Q}]}(Q_{mid})-\frac{1}{2}\right]\left(Q_{mid}-\frac{\overline{Q}+\underline{Q}}{2}\right) = 0\\ \mathrm{RE}(Q_{med}) &= \left[T_{[\underline{Q},\overline{Q}]}(Q_{med})-\frac{1}{2}\right]\left(Q_{med}-\frac{\overline{Q}+\underline{Q}}{2}\right) = 0\\ \forall\ Q\ &\in\ \mathrm{II}\ \mathit{or}\ \mathrm{IV},\\ \mathrm{RE}(Q) &= \left[T_{[\underline{Q},\overline{Q}]}(Q)-\frac{1}{2}\right]\left(Q-\frac{\overline{Q}+\underline{Q}}{2}\right) \leq 0\end{aligned}$$

□

By proposition 1 and 2, it is proved that all the questions in I and III are less optimal and if Q_{mid} and Q_{med} coincide, the optimal point is Q_{mid} (or Q_{med}). So the questions that need to be considered reduced to the values in II or IV, i.e. between Q_{mid} and Q_{med}.

Generally, it is impossible to find global optimum by pointwise comparison. Thus, IA_1 compare RE of one question with the lower bound of RE that a sub-interval can have. By this point-interval comparison, a sub-interval can be eliminated. Before illustrating specific procedure of point-interval comparison, we need one more concept, the lower bound of RE of sub-interval.

Definition 6. (The lower bound of RE of sub-interval $[Q', Q]$, $\underline{\mathrm{RE}[Q', Q]}$)

The lower bound of RE of sub-interval $[Q', Q]$, denoted by $\underline{\mathrm{RE}[Q', Q]}$, is a lower bound that RE of questions in that sub-interval can have. $\underline{\mathrm{RE}[Q', Q]}$ is defined as

$$\begin{aligned}
\underline{\mathrm{RE}[Q', Q]} &= \left(Q' - Q_{mid}\right)\left[T_{[\underline{Q},\overline{Q}]}(Q) - \frac{1}{2}\right] \quad if \;\; Q \le Q_{mid} \\
\underline{\mathrm{RE}[Q', Q]} &= \left(Q - Q_{mid}\right)\left[T_{[\underline{Q},\overline{Q}]}(Q') - \frac{1}{2}\right] \quad if \;\; Q \ge Q_{mid}.
\end{aligned}$$

Note that $\mathrm{RE}(Q)$ is non-positive value on the interval $[Q_{med}, Q_{mid}]$. So for arbitrary $Q_1, Q_2 \in [Q_{med}, Q_{mid}]$, $|\mathrm{RE}(Q_1)| > |\mathrm{RE}(Q_2)|$ implies $\mathrm{RE}(Q_1) < \mathrm{RE}(Q_2)$. Since each factor of $\underline{\mathrm{RE}[Q', Q]}$ have the largest absolute value they can have on the sub-interval $[Q', Q]$, any RE of Q in that sub-interval never be smaller than the lower bound. Thus, if RE of a question which is not in $[Q', Q]$ is less than $\underline{\mathrm{RE}[Q', Q]}$, then this implies that questions in $[Q', Q]$ never be optimal. I illustrate the logic of point-interval comparison graphically. (Figure 3)

Illustration of point-interval comparison (Figure 3)

① Consider only the questions between Q_{med} and Q_{mid}

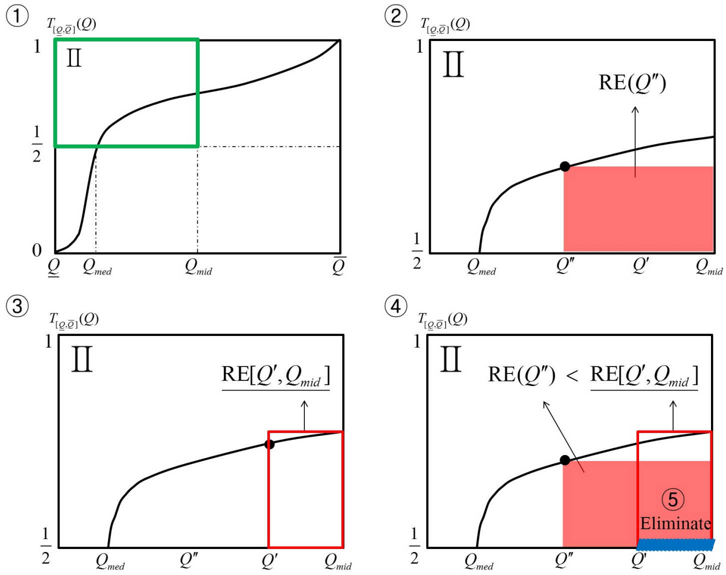


Figure 3: Illustraion of point-interval comparison

② Pick Q' and Q'' and calculate $\mathrm{RE}(Q'')$

③ Calculate $\underline{\mathrm{RE}[Q', Q_{mid}]}$

④ Compare the values of $\mathrm{RE}(Q'')$ and $\underline{\mathrm{RE}[Q', Q_{mid}]}$

⑤ Eliminate the sub-interval $[Q', Q_{mid}]$ if $\mathrm{RE}(Q'')$ is less than $\underline{\mathrm{RE}[Q', Q_{mid}]}$.

By applying this elimination procedure iteratively, the global optimum Q^* can be found numerically within the predetermined tolerance. The flowchart of specific procedures of IA_1 is depicted in Figure 4.

The global optimal question need not be unique. Thus one optimal question need to be chosen. IA_1 finds global optimum that is the nearest to Q_{mid}. Because, getting closer and closer to Q_{mid}, variance of the length of true interval of $\mathbb{Q}$ becomes smaller.[11]

[11]Like the concept of Uniformly Minimum Variance Unbiased Estimator (UMVUE), among the global optimal questions, the question which gives the minimum variance will results in stable interval estimates.

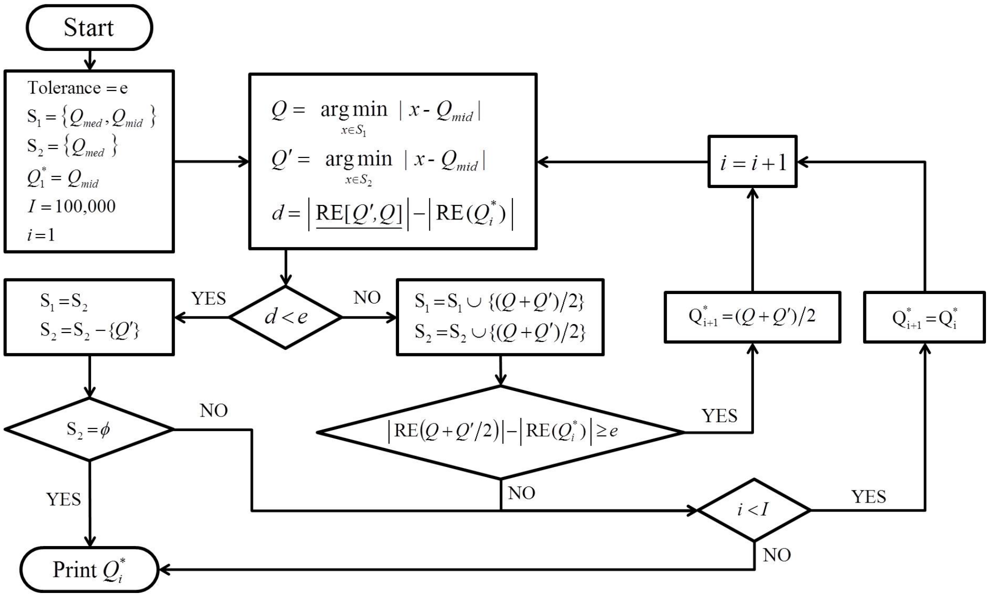


Figure 4: Flowchart of IA_1

The second sub-algorithm (IA_2)

The second sub-algorithm IA_2 is the coordinate descent algorithm. This algorithm minimizes multivariable function ELTI($\mathbb{Q}$) by sequentially finding optimal question in $\mathbb{Q}$. IA_2 utilizes IA_1 as a method for finding the optimal questions.

This algorithm makes ELTI($\mathbb{Q}$) smaller for each iteration.

Initial $\mathbb{Q}^{NM\_1} = \{Q^{1,1}, \cdots, Q^{1,N}\}$

For an $i \in 2, \cdots, N-1$

$$Q^{1,i*} = \underset{Q^{1,i} \in [Q^{1,i-1}, Q^{1,i+1}]}{\arg\min} \mathrm{ELTI}(\mathbb{Q}^{NM\_1})$$

Replace $Q^{1,i}$ in $\mathbb{Q}^{NM\_1}$ with $Q^{1,i*}$ and let this set $\mathbb{Q}'^{NM\_1}$

Then, $\mathrm{ELTI}(\mathbb{Q}^{NM\_1}) \geq \mathrm{ELTI}(\mathbb{Q}'^{NM\_1})$

This process is illustrated in Figure 5. IA_2 sequentially improves $\mathrm{ELTI}(\mathbb{Q}^{NM\_1})$ and the whole sequential process from $i = 2$ to $i = N - 1$ is called a cycle.

Flow chart of the whole algorithm of IA is depicted in Figure 6.

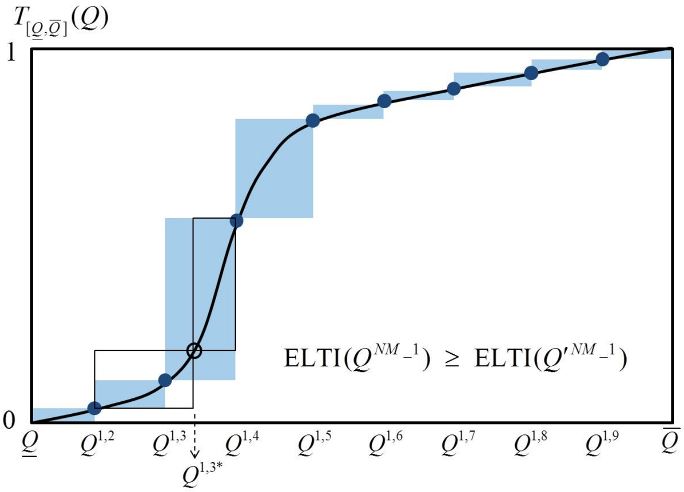


Figure 5: The process of IA_2 when i=3

In NM_1, the initial $\mathbb{Q}$ is the set of questions which equally divides $[\underline{Q}, \overline{Q}]$. The order of $\mathbb{Q}$, $|\mathbb{Q}|$, is same as the maximum number of questions to be asked. Let this initial set $\mathbb{Q}$ for constructing $\mathbb{Q}^{NM\_1}$ be $\mathbb{Q}^{ED\_1}$. [12] Let subscript j of $\mathbb{Q}_j^{NM\_1}$ denote the number of cycles which were applied to $\mathbb{Q}^{ED\_1}$. If there is no improvement of ELTI($\mathbb{Q}_j^{NM\_1}$) after one cycle, this implies that a stationary point is reached. The final set $\mathbb{Q}_j^{NM\_1}$ becomes $\mathbb{Q}^{NM\_1}$.

4.3.2 Using the information that is attainable and emerged during experiment (NM_2)

As we saw in 4.3.1, NM_1 generates more efficient $\mathbb{Q}$ than $\mathbb{Q}^{ED}$. But, as the structure of NM_1 is same with MPL, NM_1 also can not update the information emerged during experiment. By changing the structure, this new information also can be used to construct $\mathbb{Q}$. And this structural change results

[12] ED is the abbreviation of "Equally Divided"

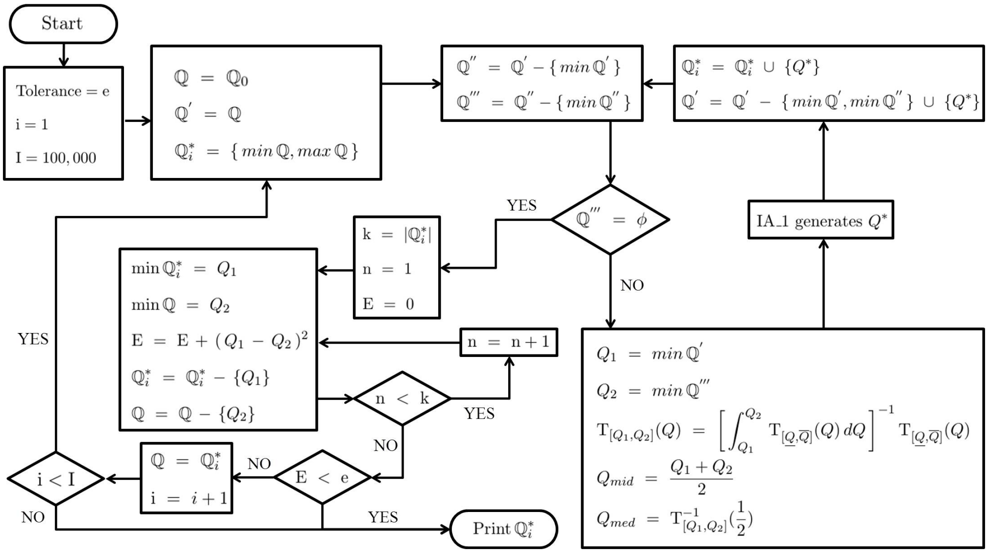


Figure 6: Flowchart of IA

in more efficient $\mathbb{Q}$ than $\mathbb{Q}^{NM\_1}$.

The $\mathbb{Q}$ and $\mathbb{S}$ of NM_2 with N questions are as follows.

$$\begin{aligned}\mathbb{Q}^{NM\_2} &= \{\, Q_0^{N,1},\, Q^{N,1}_{\sum_{k=0}^{N-2} 2^k},\, Q^{1,1},\, Q_j^{i,1} \mid i = 2, \cdots, N-1, \\ &\qquad j = 0, \cdots, \sum_{k=0}^{i-2} 2^k, Q_0^{N,1} = \underline{Q},\, Q^{N,1}_{\sum_{k=0}^{N-2} 2^k} = \overline{Q}\,\}\end{aligned}$$

$$\mathbb{S}^{NM\_2} = \left\{ \begin{pmatrix} Q^{1,1} \\ Q_0^{2,1} \\ \vdots \\ Q_0^{N-2,1} \\ Q_0^{N-1,1} \\ Q_0^{N,1} \end{pmatrix}, \begin{pmatrix} Q^{1,1} \\ Q_0^{2,1} \\ \vdots \\ Q_0^{N-2,1} \\ Q_0^{N-1,1} \\ i_1 \end{pmatrix}, \begin{pmatrix} Q^{1,1} \\ Q_0^{2,1} \\ \vdots \\ Q_0^{N-2,1} \\ Q_1^{N-1,1} \\ i_2 \end{pmatrix}, \begin{pmatrix} Q^{1,1} \\ Q_0^{2,1} \\ \vdots \\ Q_0^{N-2,1} \\ Q_1^{N-1,1} \\ i_3 \end{pmatrix}, \right.$$

$$\ldots, \begin{pmatrix} Q^{1,1} \\ Q_1^{2,1} \\ \vdots \\ Q^{N-2,1}_{\sum_{j=0}^{N-4} 2^j} \\ Q^{N-1,1}_{\sum_{j=0}^{N-3} 2^j - 1} \\ i_{\sum_{j=0}^{N-2} 2^j - 3} \end{pmatrix}, \begin{pmatrix} Q^{1,1} \\ Q_1^{2,1} \\ \vdots \\ Q^{N-2,1}_{\sum_{j=0}^{N-4} 2^j} \\ Q^{N-1,1}_{\sum_{j=0}^{N-3} 2^j - 1} \\ i_{\sum_{j=0}^{N-2} 2^j - 2} \end{pmatrix}, \begin{pmatrix} Q^{1,1} \\ Q_1^{2,1} \\ \vdots \\ Q^{N-2,1}_{\sum_{j=0}^{N-4} 2^j} \\ Q^{N-1,1}_{\sum_{j=0}^{N-3} 2^j} \\ i_{\sum_{j=0}^{N-2} 2^j - 1} \end{pmatrix}, \begin{pmatrix} Q^{1,1} \\ Q_1^{2,1} \\ \vdots \\ Q^{N-2,1}_{\sum_{j=0}^{N-4} 2^j} \\ Q^{N-1,1}_{\sum_{j=0}^{N-3} 2^j} \\ Q^{N,1}_{\sum_{j=0}^{N-2} 2^j} \end{pmatrix} \Bigg\}$$

General forms of $\mathbb{Q}^{NM\_2}$ and $\mathbb{S}^{NM\_2}$ look quite complicated. So, I present a simple example.

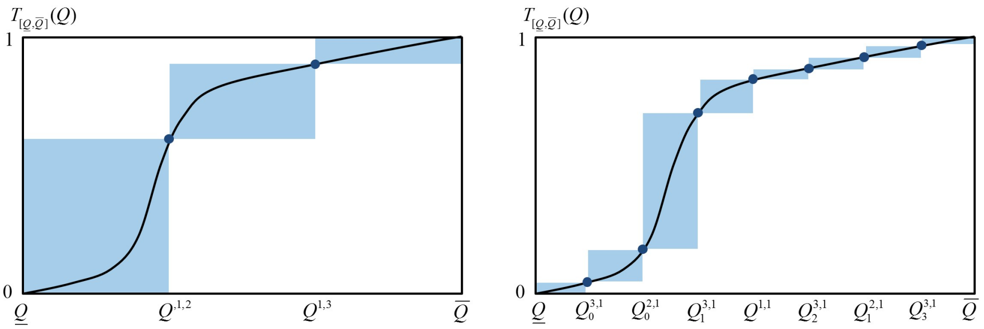
Example (The $\mathbb{Q}$ and $\mathbb{S}$ of NM_2, R $= 4$ and S $= 4$)

$$\begin{aligned} \mathbb{Q}^{NM\_2} &= \{\, Q^{1,1}, Q_0^{2,1}, Q_1^{2,1}, Q_0^{3,1}, Q_1^{3,1}, Q_2^{3,1}, Q_3^{3,1}, Q_0^{4,1}, Q_7^{4,1} \\ &\qquad \mid Q_0^{4,1} = \underline{Q}, Q_7^{4,1} = \overline{Q}\,\} \end{aligned}$$

$$\begin{aligned} \mathbb{S}^{NM\_2} &= \left\{ \begin{pmatrix} Q^{1,1} \\ Q_0^{2,1} \\ Q_0^{3,1} \\ Q_0^{4,1} \end{pmatrix}, \begin{pmatrix} Q^{1,1} \\ Q_0^{2,1} \\ Q_0^{3,1} \\ i_1 \end{pmatrix}, \begin{pmatrix} Q^{1,1} \\ Q_0^{2,1} \\ Q_1^{3,1} \\ i_2 \end{pmatrix}, \begin{pmatrix} Q^{1,1} \\ Q_0^{2,1} \\ Q_1^{3,1} \\ i_3 \end{pmatrix} \right. \\ &\qquad \left. , \begin{pmatrix} Q^{1,1} \\ Q_1^{2,1} \\ Q_2^{3,1} \\ i_4 \end{pmatrix}, \begin{pmatrix} Q^{1,1} \\ Q_1^{2,1} \\ Q_2^{3,1} \\ i_5 \end{pmatrix}, \begin{pmatrix} Q^{1,1} \\ Q_1^{2,1} \\ Q_3^{3,1} \\ i_6 \end{pmatrix}, \begin{pmatrix} Q^{1,1} \\ Q_1^{2,1} \\ Q_3^{3,1} \\ Q_7^{4,1} \end{pmatrix} \right\} \end{aligned}$$

Compared to $\mathbb{S}^{NM\_1}$, $\mathbb{S}^{NM\_2}$ consists of $\sum_{j=0}^{N-2} 2^j + 1$ matrices where N is the number of stages. This difference comes from the use of history. Each matrix in $\mathbb{S}^{NM\_2}$ denotes a series of questions that is presented to a subject according to their previous answers. Note that for a subject who has θ in inner intervals, which is not the first and last finite intervals, only N-1 questions are needed. Thus, a concept of the maximum number of questions rather than just the number of questions is appropriate for NM_2.

The use of history makes it possible to partition $[\underline{Q}, \overline{Q}]$ to much smaller length of intervals with same maximum number of questions than the methods that do not use the history. This is illustrated in Figure 7.



(a) 1 - stage, maximum 4 - questions

(b) 4 - stages, maximum 4 - questions, history used

Figure 7: The effect of using the history on ELTI (maximum 4-questions)

What's left to construct $\mathbb{Q}^{NM\_2}$ is to select value of each question. In NM_2, the initial $\mathbb{Q}$ is the set of questions which equally divides $[\underline{Q}, \overline{Q}]$. But not like a $\mathbb{Q}^{ED\_1}$, the order of $\mathbb{Q}$, $|\mathbb{Q}|$, is $\sum_{i=0}^{N-2} 2^i + 2$ where N is the maximum number of questions. Let this initial $\mathbb{Q}$ be $\mathbb{Q}^{ED\_2}$ and subscript j of $\mathbb{Q}_j^{NM\_2}$ denote the number of cycles which were applied to $\mathbb{Q}^{ED\_2}$. If there is no improvement of ELTI($\mathbb{Q}_j^{NM\_2}$) after one cycle, this implies that a stationary point is reached. The final set $\mathbb{Q}_j^{NM\_2}$ becomes $\mathbb{Q}^{NM\_2}$.

4.4 Conversion of $\mathbb{Q}^{NM}$ to specific binary economic decision problems

After $\mathbb{Q}^{NM}$ ($\mathbb{Q}^{NM\_1}$ or $\mathbb{Q}^{NM\_2}$) is constructed, this need to be converted to specific binary economic decision problems. When converting, an experimenter should be cautious, because strategic behavior might occur under $\mathbb{Q}^{NM\_2}$ which uses history. In this section, I first explain general converting procedure and then introduce a rule for conversion that prevents strategic behavior.

4.4.1 General converting procedure

The form of binary economic decision problems depends on which parametric model $f(\mathbf{X} \mid \theta)$ an experimenter assumes.

For each value Q in $\mathbb{Q}$, select two vectors $\mathbf{X_0^Q}$ and $\mathbf{X_1^Q}$ such that

$$\begin{aligned} f(\mathbf{X_0^Q} \mid Q) &= f(\mathbf{X_1^Q} \mid Q) \\ Q > \theta \Leftrightarrow f(\mathbf{X_0^Q} \mid \theta) &> f(\mathbf{X_1^Q} \mid \theta) \\ Q < \theta \Leftrightarrow f(\mathbf{X_0^Q} \mid \theta) &< f(\mathbf{X_1^Q} \mid \theta) \end{aligned}$$

In fact, there might be infinite number of vectors that satisfy above constraints. An experimenter should choose one vector among the available vectors considering some criterions, for example, total budget of experiment, front-end delay, incentive compatibility rule and so on.

Then, converted binary economic decision problems for Q is a choice problem between the vectors $\mathbf{X_0^Q}$ and $\mathbf{X_1^Q}$.

$$\mathbf{Q_1}.\quad \mathbf{X_0^{Q_1}} \quad vs. \quad \mathbf{X_1^{Q_1}}$$

$$\mathbf{Q_2}.\quad \mathbf{X_0^{Q_2}} \quad vs. \quad \mathbf{X_1^{Q_2}}$$

$$\vdots$$

$$\mathbf{Q_N}.\quad \mathbf{X_0^{Q_N}} \quad vs. \quad \mathbf{X_1^{Q_N}}$$

Example (HL's Experiment for risk preference elicitation)

In HL's experiment, θ is risk preference parameter. $f(\mathbf{X}\,|\,\theta)$ is assumed to be Constant Relative Risk Aversion (CRRA) utility function and follow Expected Utility model. $\mathbf{X}$ is a lottery that consists of values of monetary payoffs $(x_1,\, x_2)$ and corresponding probabilities $(p_1,\, p_2)$.

$$\theta \in \mathbb{R}$$

$$\begin{aligned} \mathbf{X} &= (x_1,\, x_2;\, p_1,\, p_2) \\ f(\mathbf{X} \mid \theta) &= p_1 \cdot \frac{{x_1}^{1-\theta}}{1-\theta} + p_2 \cdot \frac{{x_2}^{1-\theta}}{1-\theta} \end{aligned}$$

Let $\mathbb{Q}$ be constructed as

$$\mathbb{Q} = \{-0.95,\, -0.15,\, 0.41,\, 0.97,\, 1.37\}$$

For each value Q in $\mathbb{Q}$, select two vectors $\mathbf{X_0^Q}$ and $\mathbf{X_1^Q}$ such that the above constraints of the general converting procedure are satisfied.[13] The selected vectors are presented in Table 1.

The converted binary economic decision problems are as follows

$\mathbf{Q_1}$. 0.2 \$3.85 0.8 \$0.10 *vs.* 0.2 \$2.00 0.8 \$1.60

$\mathbf{Q_2}$. 0.4 \$3.85 0.6 \$0.10 *vs.* 0.4 \$2.00 0.6 \$1.60

$\mathbf{Q_3}$. 0.6 \$3.85 0.4 \$0.10 *vs.* 0.6 \$2.00 0.4 \$1.60

$\mathbf{Q_4}$. 0.8 \$3.85 0.2 \$0.10 *vs.* 0.8 \$2.00 0.2 \$1.60

$\mathbf{Q_5}$. 0.9 \$3.85 0.1 \$0.10 *vs.* 0.9 \$2.00 0.1 \$1.60

[13] These are some of the values which are used in Holt and Laury(2002)

Q	$\mathbf{X}^Q$	x_1	x_2	p_1	p_2
-0.95	$\mathbf{X}_0^{-0.95}$	\$3.85	\$0.10	0.2	0.8
-0.95	$\mathbf{X}_1^{-0.95}$	\$2.00	\$1.60	0.2	0.8
-0.15	$\mathbf{X}_0^{-0.15}$	\$3.85	\$0.10	0.4	0.6
-0.15	$\mathbf{X}_1^{-0.15}$	\$2.00	\$1.60	0.4	0.6
0.41	$\mathbf{X}_0^{0.41}$	\$3.85	\$0.10	0.6	0.4
0.41	$\mathbf{X}_1^{0.41}$	\$2.00	\$1.60	0.6	0.4
0.97	$\mathbf{X}_0^{0.97}$	\$3.85	\$0.10	0.8	0.2
0.97	$\mathbf{X}_1^{0.97}$	\$2.00	\$1.60	0.8	0.2
1.37	$\mathbf{X}_0^{1.37}$	\$3.85	\$0.10	0.9	0.1
1.37	$\mathbf{X}_1^{1.37}$	\$2.00	\$1.60	0.9	0.1

Table 1: Converted vectors for risk preference questions

Example (Experiment for time preference elicitation)

In this experiment, θ is time preference parameter. $f(\mathbf{X}\,|\,\theta)$ is assumed to be inear utility function and have Exponential Discount Utility model. $\mathbf{X}$ consists of values of monetary payoff (x) and time (t, weeks) at which x is paid.

$$\begin{aligned} &\theta \in \mathbb{R} \\ \mathbf{X} &= (x\,;\,t) \\ f(\mathbf{X} \mid \theta) &= \frac{1}{(1+\theta)^t} \cdot x \end{aligned}$$

Let $\mathbb{Q}$ be constructed as

$$\mathbb{Q} = \{\,0.10,\, 0.15,\, 0.20,\, 0.25,\, 0.30\,\}$$

For each value Q in $\mathbb{Q}$, select two vectors $\mathbf{X_0^Q}$ and $\mathbf{X_1^Q}$ such that the above constraints of the general converting procedure are satisfied. The selected vectors are presented in Table 2.

Q	$\mathbf{X}^Q$	x_1	x_2	**Q**	$\mathbf{X}^Q$	x_1	x_2
0.10	$\mathbf{X}_0^{0.10}$	\$33.13	6	0.20	$\mathbf{X}_1^{0.20}$	\$30.00	0
0.10	$\mathbf{X}_1^{0.10}$	\$30.00	0	0.25	$\mathbf{X}_0^{0.25}$	\$38.32	6
0.15	$\mathbf{X}_0^{0.15}$	\$34.79	6	0.25	$\mathbf{X}_1^{0.25}$	\$30.00	0
0.15	$\mathbf{X}_1^{0.15}$	\$30.00	0	0.30	$\mathbf{X}_0^{0.30}$	\$40.20	6
0.20	$\mathbf{X}_0^{0.20}$	\$36.52	6	0.30	$\mathbf{X}_1^{0.30}$	\$30.00	0

Table 2: Converted vectors for time preference questions

Converted binary economic decision problems are as follows

$\mathbf{Q_1}$. 6 *weeks later* \$33.13 *vs.* *today* \$30.00

$\mathbf{Q_2}$. 6 *weeks later* \$34.79 *vs.* *today* \$30.00

$\mathbf{Q_3}$. 6 *weeks later* \$36.52 *vs.* *today* \$30.00

$\mathbf{Q_4}$. 6 *weeks later* \$38.32 *vs.* *today* \$30.00

$\mathbf{Q_5}$. 6 *weeks later* \$40.20 *vs.* *today* \$30.00

4.4.2 A rule for preventing strategic behavior

In NM_2, subject's previous choices have influence on the values of the next questions. Thus, if there is a chance to achieve more benefit by deceiving their true economic preference, they would not act truthfully. Under such an experiment design, elicited T^θ and estimated $\hat{\theta}$ may be biased. So, when using history, additional instrument which prevents strategic behavior is needed. I introduce a rule for converting procedure which prevents strategic behavior.

To explain this rule, I define one more concept, the order of T^θ.

Definition 7. (The order of T^{θ})

The order of T^{θ} is non-negative integer and defined as

$$\begin{aligned}&\text{For arbitrary } \mathrm{T}^{\theta} \text{ and } \mathrm{T}^{\theta\prime}, \mathrm{T}^{\theta} \text{ has smaller order than } \mathrm{T}^{\theta\prime}\\ &\Leftrightarrow\ \forall\, \theta \in \mathrm{T}^{\theta}\ \ and\ \ \theta' \in \mathrm{T}^{\theta\prime},\ \ \theta \leq \theta'\end{aligned}$$

The smallest order is 0 and subscript o of T^{θ}_{o} denotes the order of T^{θ}.

$$\mathrm{T}^{\theta}_{o}$$

This concept of the order of T^{θ} is illustrated in Figure 8. In the Figure, each dot denotes binary economic decision problems and the numbered lines from the dot means binary options for that problem.

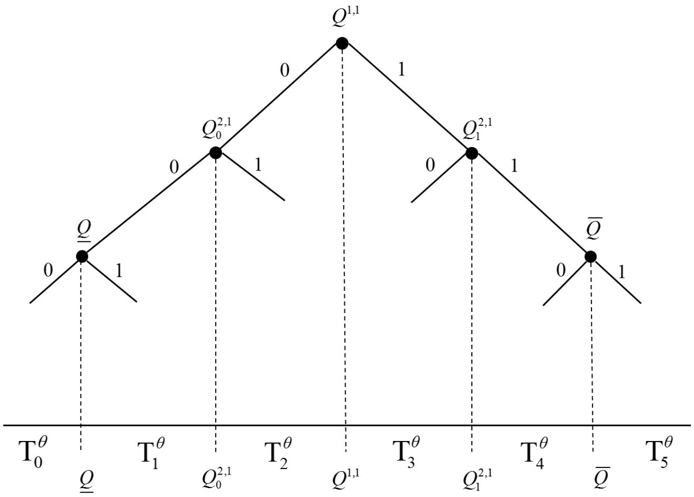


Figure 8: The order of T^{θ}

A rule for preventing strategic behavior[14]

The particular values of converted binary economic decision problems should be selected such that a subject who has θ in T^{θ}_{o} must have highest utility from the choices which lead to T^{θ}_{o}.

[14]Let this rule be called Incentive compatibility rule.

Under the converted binary economic decision problems that satisfy this rule, there is no incentive for subjects to deceive their true preference. Because, they have the highest utility when they choose options which lead to T_o^θ that contains their true value θ.

Example (Applying the rule for risk preference elicitation)

As in the Figure 8, if the $\mathbb{Q}$ of the above example (HL's risk preference experiment) are presented sequentially, each stage question and T_o^θ for each order o are as follows.

$$\mathbb{Q} = \{-0.95, -0.15, 0.41, 0.97, 1.37\}$$

$$\begin{aligned} Q^{1,1} &= 0.41 \\ Q_0^{2,1} &= -0.15 \\ Q_1^{2,1} &= 0.97 \\ Q_0^{3,1} &= -0.95 \\ Q_3^{3,1} &= 1.37 \end{aligned}$$

$$\begin{aligned} \mathrm{T}_0^\theta &= [-\infty, -0.95] \\ \mathrm{T}_1^\theta &= [-0.95, -0.15] \\ \mathrm{T}_2^\theta &= [-0.15, 0.41] \\ \mathrm{T}_3^\theta &= [0.41, 0.97] \\ \mathrm{T}_4^\theta &= [0.97, 1.37] \\ \mathrm{T}_5^\theta &= [1.37, \infty] \end{aligned}$$

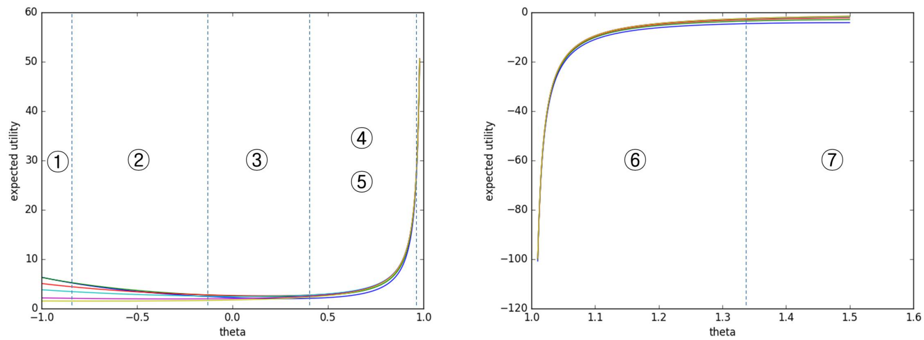
In this risk preference experiment, only one option which is randomly selected among the chosen options will be realized. Thus, the expected utilities of the choices that lead to each possible T^θ are as follows.

$$
\begin{aligned}
\mathrm{T}_0^\theta : \mathrm{EU\_T}_0^\theta &= \frac{1}{3}\left[f(X_0^{0.41}|\theta) + f(X_0^{-0.15}|\theta) + f(X_0^{-0.95}|\theta)\right] \\
\mathrm{T}_1^\theta : \mathrm{EU\_T}_1^\theta &= \frac{1}{3}\left[f(X_0^{0.41}|\theta) + f(X_0^{-0.15}|\theta) + f(X_1^{-0.95}|\theta)\right] \\
\mathrm{T}_2^\theta : \mathrm{EU\_T}_2^\theta &= \frac{1}{2}\left[f(X_0^{0.41}|\theta) + f(X_1^{-0.15}|\theta)\right] \\
\mathrm{T}_3^\theta : \mathrm{EU\_T}_3^\theta &= \frac{1}{2}\left[f(X_1^{0.41}|\theta) + f(X_0^{0.97}|\theta)\right] \\
\mathrm{T}_4^\theta : \mathrm{EU\_T}_4^\theta &= \frac{1}{3}\left[f(X_1^{0.41}|\theta) + f(X_1^{0.97}|\theta) + f(X_0^{1.37}|\theta)\right] \\
\mathrm{T}_5^\theta : \mathrm{EU\_T}_5^\theta &= \frac{1}{3}\left[f(X_1^{0.41}|\theta) + f(X_1^{0.97}|\theta) + f(X_1^{1.37}|\theta)\right] \\
\text{where} \quad f(\mathbf{X} \mid \theta) &= p_1 \cdot \frac{{x_1}^{1-\theta}}{1-\theta} + p_2 \cdot \frac{{x_2}^{1-\theta}}{1-\theta}
\end{aligned}
$$

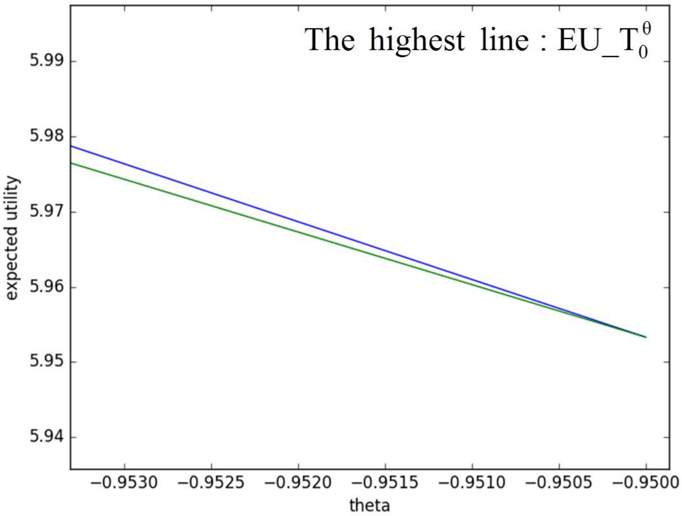
An experimenter has to find the values of vectors that satisfy the rule for general converting procedure and preventing the strategic behavior. By solving the system of equations, these values can be found. the values are as follows

1-th stage	0.6 \$3.85 0.4 \$0.10	*vs.*	0.6 \$2.00 0.4 \$1.60
2-th stage-1	0.4 \$6.62 0.6 \$0.10	*vs.*	0.4 \$4.97 0.6 \$1.60,
(if previous answer was 0)			
2-th stage-2	0.8 \$3.85 0.2 \$0.10	*vs.*	0.8 \$2.00 0.2 \$1.60,
(if previous answer was 1)			
3-th stage-1	0.2 \$7.64 0.8 \$0.10	*vs.*	0.2 \$6.86 0.8 \$1.60,
(if previous answers were (0,0))			
3-th stage-1	0.9 \$2.55 0.1 \$0.10	*vs.*	0.9 \$1.44 0.1 \$1.60,
(if previous answers were (1,1))			

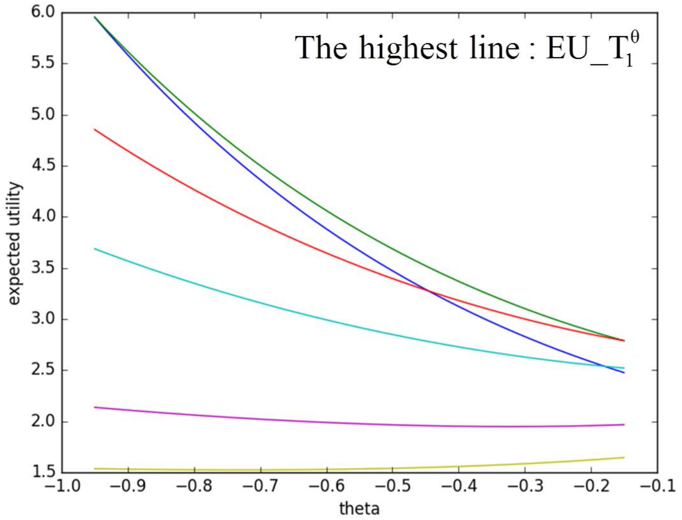
The following Figure 9 shows that the expected utility of each possible choices along the values of θ. For every θ in T^{θ}, the choices that has the highest expected utility is the options that leads to that T^{θ}. (Figure 9) So, there is no incentive to deceive their true preferences.



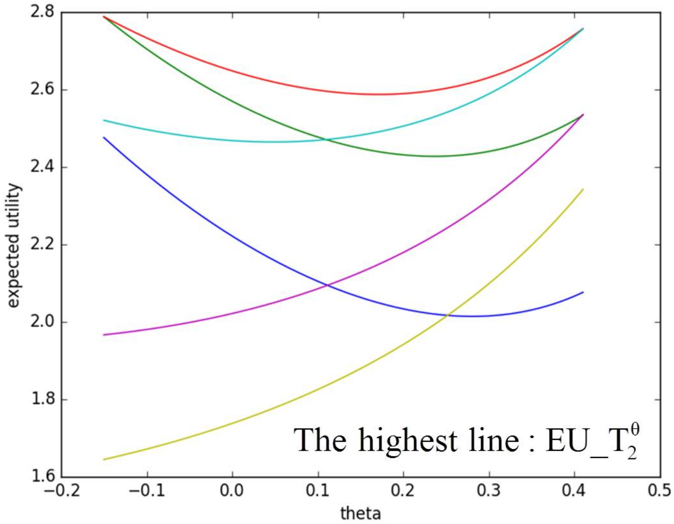
① $T_0^{\theta} = [-\infty, -0.95]$



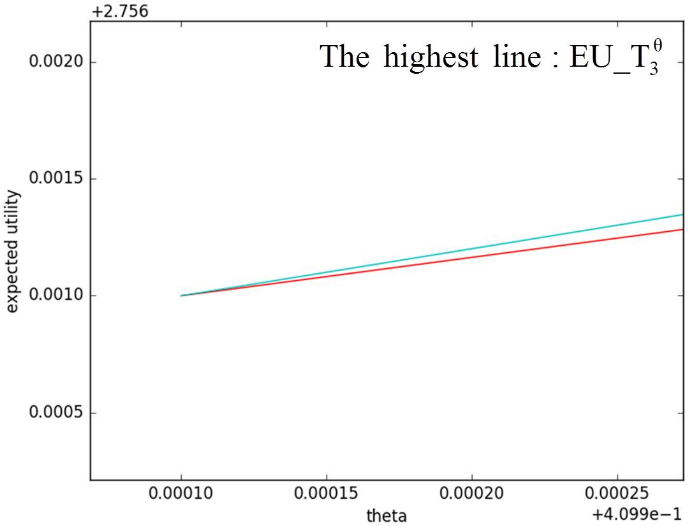
② $T_1^{\theta} = [-0.95, -0.15]$



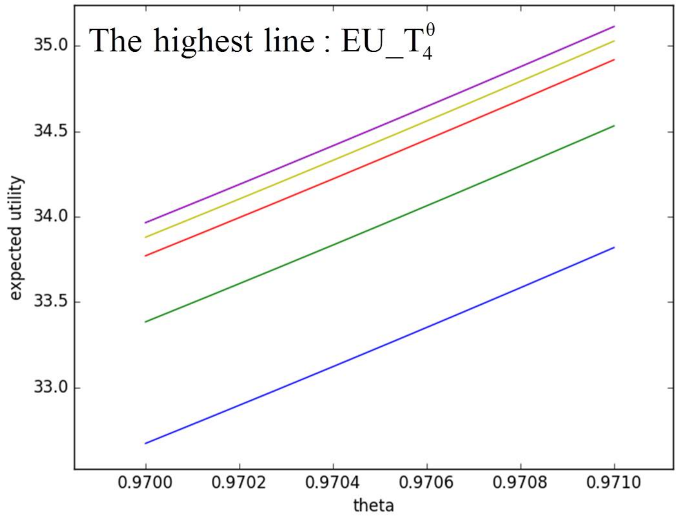
③ $T_2^{\theta} = [-0.15, 0.41]$



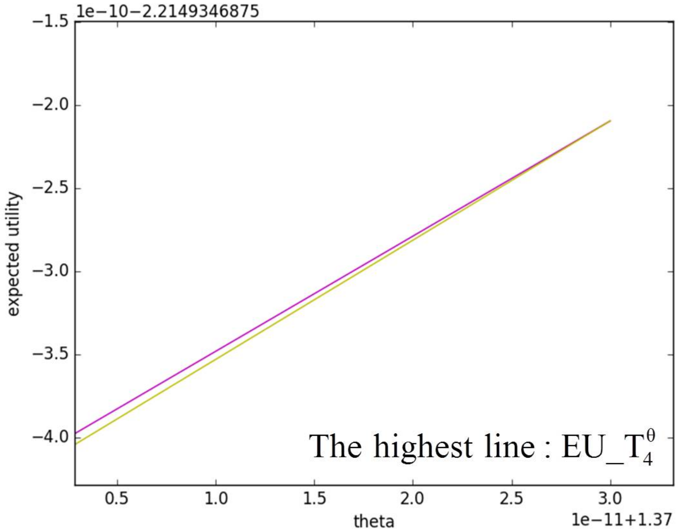
④ $T_3^{\theta} = [0.41, 0.97]$



⑤ $T_4^{\theta} = [0.97, 1.00]$



⑥ $T_4^\theta = [1.00, 1.37]$



⑦ $T_5^\theta = [1.37, 2.50]$

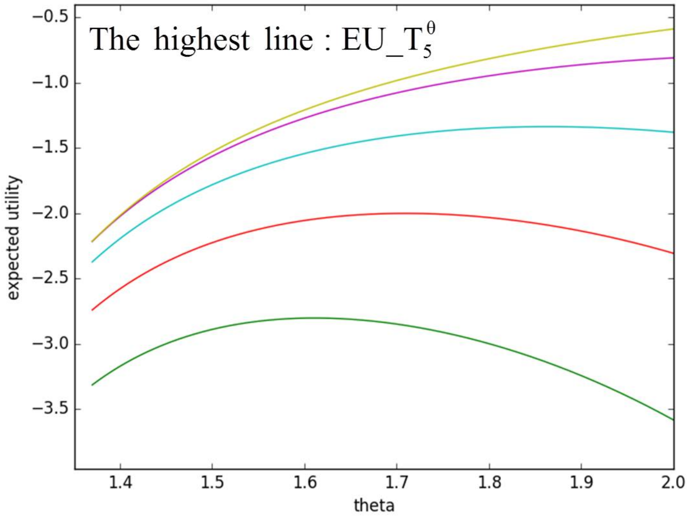


Figure 9: Expected utility at each T_o^θ

5 An Experimental Application

In this section, I present an exprimental application of NM. An experimenter, $\mathbb{E}$, is designing an experiment for risk preference elicitation using NM.

5.1 Construction of prior and posterior distribution

From the previous similar experiment, Andersen et al.(2008), $\mathbb{E}$ could get the information about the possible range of risk preference under CRRA utility function, $[-1.84, 2.21]$. Conservatively she set the possible range of risk preference little wider, $[-2, 2.5]$, and converted this information to prior distribution.

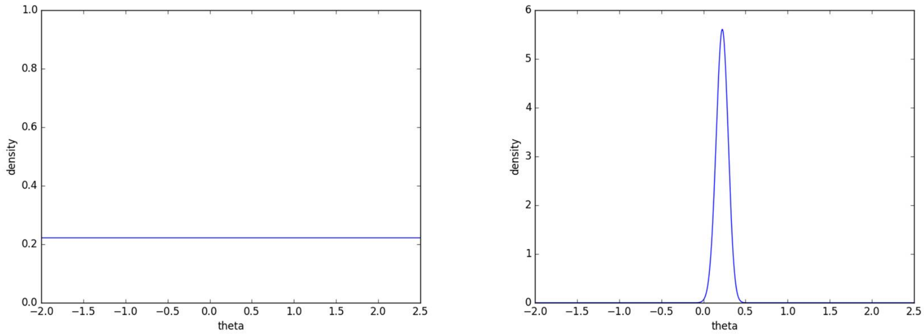
$$\pi(\theta) \quad \sim \quad uniform[-2, 2.5]$$

$\mathbb{E}$ also could attain data of the previous similar experiment from Econometrica online webpage. She used the data for the Bayes update and utilized simple likelihood function from Holt and Laury (2002).

$$\mathrm{L}(\mathrm{X}|\theta) \quad = \quad \prod_{i=1}^{10} \frac{\mathrm{EU}(\mathrm{x}_i|\theta)}{\mathrm{EU}(\mathrm{X}_0^{Q_i}|\theta) + \mathrm{EU}(\mathrm{X}_1^{Q_i}|\theta)}$$

$$\begin{aligned}
where,\ & \mathrm{X} \text{ is an answer vector, } \mathrm{X} = (x_1, x_2, \cdots, x_{10}) \\
& i = 1, 2, \cdots, 10 \\
& \mathrm{x}_i \ \in \ \{\, 0,\, 1\, \} \\
& \mathrm{X}_0^{Q_i} \ = \ (a1_i,\, a2_i\, ;\, \mathrm{p1}_i,\, \mathrm{p2}_i) \\
& \mathrm{X}_1^{Q_i} \ = \ (b1_i,\, b2_i\, ;\, \mathrm{p1}_i,\, \mathrm{p2}_i) \\
& \mathrm{EU}(\mathrm{X}_0^{Q_i}|\theta) = \mathrm{p1}_i \frac{(a1_i)^{1-\theta}}{1-\theta} + \mathrm{p2}_i \frac{(a2_i)^{1-\theta}}{1-\theta} \\
& \mathrm{EU}(\mathrm{X}_1^{Q_i}|\theta) = \mathrm{p1}_i \frac{(b1_i)^{1-\theta}}{1-\theta} + \mathrm{p2}_i \frac{(b2_i)^{1-\theta}}{1-\theta} \\
& \mathrm{EU}(\mathrm{x_i}|\theta) = \begin{cases} \mathrm{EU}(\mathrm{X}_0^{Q_i}|\theta) \ if \ \mathrm{x}_i = 0 \\ \mathrm{EU}(\mathrm{X}_1^{Q_i}|\theta) \ if \ \mathrm{x}_i = 1 \end{cases}
\end{aligned}$$

$\mathbb{E}$ used 20 subjects' choice data which have no indifference choices and consist of 10 females and 10 males. The prior and posterior distributions are depicted in Figure 10.



(a) Prior distribution

(b) Posterior distribution

Figure 10: The prior and posterior distribution

5.2 Selection of the loss level α and the maximum number of questions to be asked

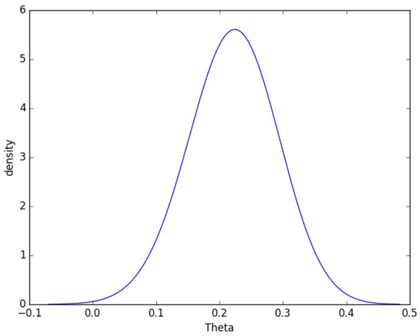
$\mathbb{E}$ set α to be 0.0001. Under this loss level, she expected 0.01% of subjects she would fail to elicit T^θ which has finite length. She decided to ask maximum 3 questions to a subject.

5.3 Construction of $\mathbb{Q}^{NM}$

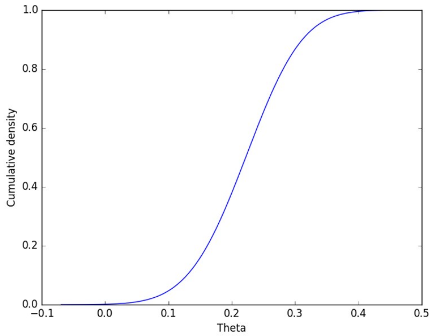
Firstly, $\mathbb{E}$ selected boundary questions under α to satisfy that the mean of the truncated posterior distribution equals to the mean of the original posterior distribution. The selected boundary questions and the values of the means are as follows.

$$\underline{Q} = -0.0695$$

$$\overline{Q} = 0.4839$$



(a) pdf of the truncated posterior distribution



(b) cdf of the truncated posterior distribution

Figure 11: The truncated pdf and cdf of posterior distribution

$$\text{The mean of the original posterior distribution} = 0.2211$$

$$\text{The mean of the truncated posterior distribution} = 0.2211$$

The truncated posterior distribution and its cdf form are depicted in Figure 11.

From $\mathbb{Q}^{ED\_1}$ which divides $[\underline{Q}, \overline{Q}]$ equally with 3 questions, IA was iteratively applied to generate $\mathbb{Q}^{NM\_1}$. And $\mathbb{Q}^{NM\_2}$ was generated from $\mathbb{Q}^{ED\_2}$ which has 5 questions by applying IA. The values of questions and structures of ED_1, NM_1, ED_2 and NM_2 are as follows.

$$\mathbb{Q}^{ED\_1} = \{-0.0695, 0.2072, 0.4839\}$$

$$\mathbb{S}^{ED\_1} = \left\{ \begin{pmatrix} -0.0695^{1,1} & 0.2072^{1,2} & 0.4839^{1,3} \end{pmatrix} \right\}$$

$$\mathbb{Q}^{NM\_1} = \{-0.0695, 0.2146, 0.4839\}$$

$$\mathbb{S}^{NM\_1} = \left\{ \begin{pmatrix} -0.0695^{1,1} & 0.2146^{1,2} & 0.4839^{1,3} \end{pmatrix} \right\}$$

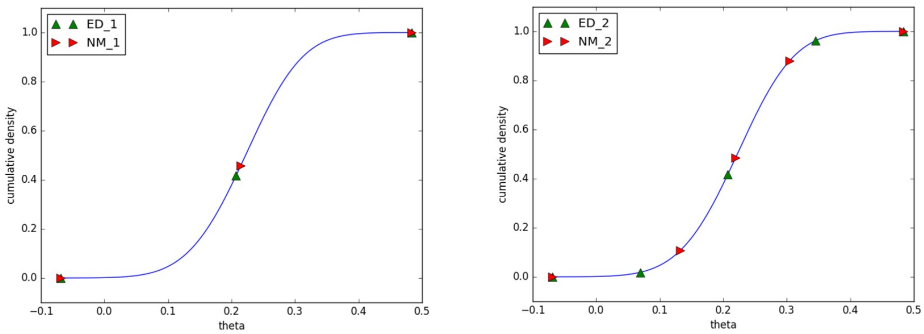
$$\mathbb{Q}^{ED\_2} = \{-0.0695, 0.0689, 0.2072, 0.3455, 0.4839\}$$

$$\mathbb{S}^{ED\_2} = \left\{ \begin{pmatrix} 0.2072^{1,1} \\ 0.0689^{2,1}_{0} \\ -0.0695^{3,1}_{0} \end{pmatrix}, \begin{pmatrix} 0.2072^{1,1} \\ 0.0689^{2,1}_{0} \\ i_1 \end{pmatrix}, \begin{pmatrix} 0.2072^{1,1} \\ 0.3455^{2,1}_{1} \\ i_2 \end{pmatrix}, \begin{pmatrix} 0.2072^{1,1} \\ 0.3455^{2,1}_{1} \\ 0.4839^{3,1}_{3} \end{pmatrix} \right\}$$

$$\mathbb{Q}^{NM\_2} = \{-0.0695, 0.1320, 0.2195, 0.3046, 0.4839\}$$

$$\mathbb{S}^{NM\_2} = \left\{ \begin{pmatrix} 0.2195^{1,1} \\ 0.1320^{2,1}_{0} \\ -0.0695^{3,1}_{0} \end{pmatrix}, \begin{pmatrix} 0.2195^{1,1} \\ 0.1320^{2,1}_{0} \\ i_1 \end{pmatrix}, \begin{pmatrix} 0.2195^{1,1} \\ 0.3046^{2,1}_{1} \\ i_2 \end{pmatrix}, \begin{pmatrix} 0.2195^{1,1} \\ 0.3046^{2,1}_{1} \\ 0.4839^{3,1}_{3} \end{pmatrix} \right\}$$

When applying IA, $\mathbb{E}$ set the acceptable tolerance of IA to be 0.0001. The number of cycles for $\mathbb{Q}^{NM\_1}$ and $\mathbb{Q}^{NM\_2}$ were 1 and 3, respectively. $\mathbb{Q}^{ED\_1}$, $\mathbb{Q}^{NM\_1}$, $\mathbb{Q}^{ED\_2}$ and $\mathbb{Q}^{NM\_2}$ are depicted in Figure 12.



(a) Questions of ED_1 and NM_1

(b) Questions of ED_2 and NM_2

Figure 12: Questions of ED_1, NM_1, ED_2 and NM_2

5.4 Conversion of $\mathbb{Q}^{NM}$ to specific binary economic decision problems.

$\mathbb{E}$ converted selected questions to binary economic decision problems considering the incentive compatibility rule. The converted binary choie questions of $\mathbb{Q}^{NM\_1}$ and $\mathbb{Q}^{NM\_2}$ are as follows.

Converted binary economic decision problems of $\mathbb{Q}^{NM\_1}$

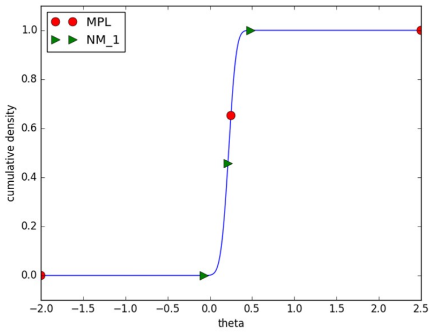
1-th stage	$Q1$.	0.2 \$7.29 0.8 \$0.10	*vs.*	0.2 \$2.00 0.8 \$1.60
	$Q2$.	0.5 \$4.06 0.5 \$0.10	*vs.*	0.5 \$2.00 0.5 \$1.60,
	$Q3$.	0.8 \$2.71 0.2 \$0.10	*vs.*	0.8 \$2.00 0.2 \$1.60,

Converted binary economic decision problems of $\mathbb{Q}^{NM\_2}$

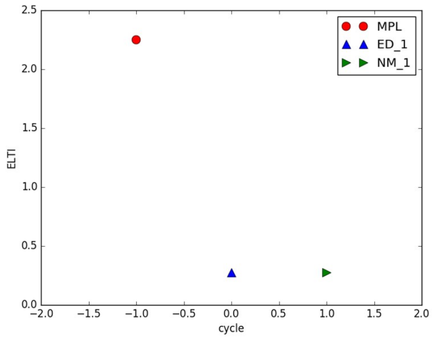
1-th stage	0.6 \$3.35 0.4 \$0.10	*vs.*	0.6 \$2.00 0.4 \$1.60
2-th stage-1	0.4 \$7.15 0.6 \$0.10	*vs.*	0.4 \$4.19 0.6 \$1.60,
	(if previous answer was 0)		
2-th stage-2	0.8 \$3.35 0.2 \$0.10	*vs.*	0.8 \$2.75 0.2 \$1.60,
	(if previous answer was 1)		
3-th stage-1	0.2 \$10.21 0.8 \$0.10	*vs.*	0.2 \$5.12 0.8 \$1.60,
	(if previous answers were (0,0))		
3-th stage-1	0.9 \$2.48 0.1 \$0.10	*vs.*	0.9 \$2.16 0.1 \$1.60,
	(if previous answers were (1,1))		

	$\mathbb{Q}^{MPL}$	$\mathbb{Q}^{ED\_1}$	$\mathbb{Q}_1^{NM\_1}$	$\mathbb{Q}^{ED\_2}$	$\mathbb{Q}_1^{NM\_2}$	$\mathbb{Q}_2^{NM\_2}$	$\mathbb{Q}_3^{NM\_2}$
questions	3	3	3	3	3	3	3
ELTI	2.2500	0.2767	0.2760	0.1383	0.1117	0.1098	0.1096

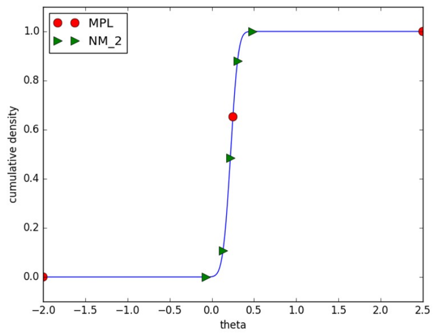
Table 3: The number of questions and ELTI of MPL, ED_1, NM_1, ED_2 and NM_2



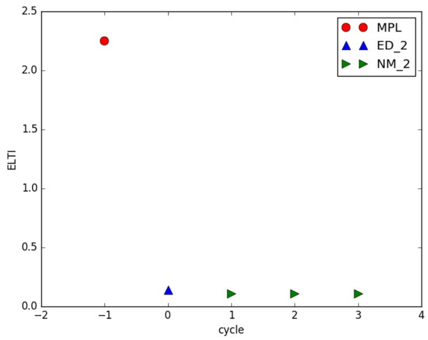
(a) Questions of MPL and NM_1



(b) ELTI of MPL, ED_1 and NM_1



(c) Questions of MPL and NM_2



(d) ELTI of MPL, ED_2 and NM_2

Figure 13: Questions and ELTI of MPL, ED_1, NM_1, ED_2 and NM_2

Table 3 and Figure 13 show the comparisons of $\mathbb{Q}$ and ELTI of MPL, ED_1, ED_2, NM_1 and NM_2. Note that NM_2 has almost 20 times smaller ELTI than MPL with same number of questions.

6 Conclusion

In this paper, I suggested a New Method (NM) which can elicit smaller interval estimates of economic preferences than the conventional method, Multiple Price List design (MPL), with the same or smaller number of questions.

One of the main ideas of NM is finding the questions which make the Expected Length of True Interval (ELTI) as small as possible. To find these questions, prior and posterior distribution that is constructed with attainable information are used. Under these distributions, the Improvement Algorithm (IA) is iteratively applied. The application of IA results in the set of questions which has smaller ELTI than the previous ones.

In addition to using the attainable information, newly emerged information during experiment is also used by sequential structure. This structure presents only one question at each stage and each question depends on the subject's previous answers. Under this structure, strategic behavior of subjects might be of concern because they can choose next questions by changing their choices. Thus, I suggested an incentive compatibility rule that prevents strategic behavior.

After constructing the set of questions, these questions should be converted to specific binary economic decision problems. When doing this, rules such as budget constraint and the incentive compatibility need to be considered.

As I showed in this paper, NM is more efficient than MPL. But this is only true when prior and posterior distributions are well constructed. Thus, additional research for construction of prior and posterior distribution need to be done.

Everything has a price. NM seems little complicated at first, but this could be considered as a price for more efficient experiment. I wish experimenters will be able to make a more efficient and less burdensome experiment by applying the NM.

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A The relation between the length of interval estimate and the risk of getting wrong point estimate.

In this appendix, I will show you How the length of interval estimate is related to the risk of getting wrong point estimate.

First of all, we need to evaluate the risk. I evaluate the risk using a quadratic loss function and the Bayes risk.

$$\begin{aligned} L(\theta,\hat{\theta}) &= (\theta-\hat{\theta})^2 \\ R(\hat{\theta}) &= \mathrm{E}[(\theta-\hat{\theta})^2] \;=\; \int_{\mathrm{T}^\theta} (y-\hat{\theta})^2\, t_{\mathrm{T}^\theta}(y|\mathrm{X})\, dy. \end{aligned}$$

Decomposition of $R(\hat{\theta})$ gives us useful information.

Decomposition of $R(\hat{\theta})$

$$\begin{aligned} R(\hat{\theta}) &= \mathrm{E}\left[(\hat{\theta}-\theta)^2\right] \\ &= \mathrm{E}\left[(\hat{\theta}-\mathrm{E}(\theta)+\mathrm{E}(\theta)-\theta)^2\right] \\ &= \mathrm{E}\left[(\hat{\theta}-\mathrm{E}(\theta))^2+2(\hat{\theta}-\mathrm{E}(\theta))(\mathrm{E}(\theta)-\theta)+(\mathrm{E}(\theta)-\theta)^2\right] \\ &= \mathrm{E}\left[(\hat{\theta}-\mathrm{E}(\theta))^2\right]+2\mathrm{E}\left[(\hat{\theta}-\mathrm{E}(\theta))(\mathrm{E}(\theta)-\theta)\right]+\mathrm{E}\left[(\mathrm{E}(\theta)-\theta)^2\right] \\ &= \mathrm{E}\left[(\hat{\theta}-E(\theta))^2\right]+2(\hat{\theta}-\mathrm{E}(\theta))\mathrm{E}\left[(\mathrm{E}(\theta)-\theta)\right]+\mathrm{E}\left[(\mathrm{E}(\theta)-\theta)^2\right] \\ &= (\hat{\theta}-\mathrm{E}(\theta))^2+\mathrm{E}\left[(\mathrm{E}(\theta)-\theta)^2\right] \\ &= \mathrm{Bias}(\hat{\theta},\mathrm{E}(\theta))^2+\mathrm{Var}(\theta) \qquad \left(\mathrm{Bias}(\hat{\theta},\mathrm{E}(\theta)) \;:=\; \hat{\theta}-\mathrm{E}(\theta)\right) \end{aligned}$$

This shows that the Bayes estimator $\hat{\theta}^{Bayes}$ that minimize the Bayes risk is $\mathrm{E}(\theta)$.

$$\begin{aligned} \hat{\theta}^{Bayes} &= \underset{\hat{\theta}\in\mathrm{T}^\theta}{\arg\min}\, R(\hat{\theta}) \;=\; \underset{\hat{\theta}\in\mathrm{T}^\theta}{\arg\min}\, (\hat{\theta}-\mathrm{E}(\theta))^2+\mathrm{E}\left[(\mathrm{E}(\theta)-\theta)^2\right] \\ &= \mathrm{E}(\theta) \end{aligned}$$

Therefore, after eliciting T^θ, $\mathrm{E}(\theta)$ is the optimal choice for minimizing the risk.

The variance term $Var(\theta)$ of the decomposition of $R(\hat{\theta})$ can be bounded by an upper bound. The following inequality gives an upper bound on $Var(\theta)$.

The Bhatia-Davis inequality[15]

The Bhatia-Davis inequality is an upper bound on the variance of any bounded probability distribution on the real line. Suppose that the truncated distribution $t_{\mathrm{T}^\theta}(\theta|X)$ has minimum Q_m, maximum Q_M, and expected value $\mathrm{E}(\theta)$. Then the inequality says

$$\mathrm{Var}(\theta) \quad \leq \quad (Q_M - \mathrm{E}(\theta))(\mathrm{E}(\theta) - Q_m)$$

Equality holds precisely if all of the probability is concetrated at the end points Q_m and Q_M.

Decomposition of $R(\hat{\theta})$ and the Bhatia-Davis inequality show that broader length of interval estimate is related to larger upper bound of the Bayes risk.

We can also consider the Bayes risk for all the possible cases of T^θ. Let this concept be the total Bayes risk.

Definition 8. (The total Bayes risk, $TR(\hat{\theta})$)

There will be an estimate for each possible T^θ which has finite length. Thus for all the possible T^θ, a vetor of estimates $\hat{\theta}$ can be considered. For that vector of estimates, $\hat{\theta} = (\hat{\theta}_1, \ \cdots, \ \hat{\theta}_{|\mathbb{Q}|-1})$, the total Bayes risk is defined as

$$TR(\hat{\theta}) \quad = \quad \sum_{i=1}^{|\mathbb{Q}|-1} \left[T_{[\underline{Q},\overline{Q}]}(Q_{i+1}|\mathrm{X}) - T_{[\underline{Q},\overline{Q}]}(Q_i|\mathrm{X}) \right] R(\hat{\theta}_i)$$

[15]The description of the Bhatia-Davis inequality is from Wikipedia.

$$Q_i \;\in\; \mathbb{Q} \;\; for \;\; i = 1, \cdots, |\mathbb{Q}|$$

$$Q_{i+1} > Q_i \;\; for \;\; i = 1, \cdots, |\mathbb{Q}| - 1$$

As the above decomposition of $R(\hat{\theta})$, we can also decompose $TR(\hat{\theta})$. This decomposition gives us the relation between ELTI and $TR(\hat{\theta})$.

Decomposition of $TR(\hat{\theta})$

$$
\begin{aligned}
& TR(\hat{\theta}) \\
= \;& \sum_{i=1}^{|\mathbb{Q}|-1} \Big[T_{[\underline{Q},\overline{Q}]}(Q_{i+1}|\mathrm{X}) - T_{[\underline{Q},\overline{Q}]}(Q_i|\mathrm{X})\Big] \, R(\hat{\theta}_i) \\
= \;& \sum_{i=1}^{|\mathbb{Q}|-1} \Big[T_{[\underline{Q},\overline{Q}]}(Q_{i+1}|\mathrm{X}) - T_{[\underline{Q},\overline{Q}]}(Q_i|\mathrm{X})\Big] \Big(\mathrm{Bias}(\hat{\theta}, \mathrm{E}(\theta))^2 + \mathrm{Var}(\theta)\Big) \\
= \;& \sum_{i=1}^{|\mathbb{Q}|-1} \Big[T_{[\underline{Q},\overline{Q}]}(Q_{i+1}|\mathrm{X}) - T_{[\underline{Q},\overline{Q}]}(Q_i|\mathrm{X})\Big] \, \mathrm{Var}(\theta) \quad (\because \hat{\theta}^{Bayes} \;=\; \mathrm{E}(\theta)\,) \\
\leq \;& \sum_{i=1}^{|\mathbb{Q}|-1} \Big[T_{[\underline{Q},\overline{Q}]}(Q_{i+1}|\mathrm{X}) - T_{[\underline{Q},\overline{Q}]}(Q_i|\mathrm{X})\Big] \, (Q_{i+1} - \mathrm{E}(\theta))(\mathrm{E}(\theta) - Q_i) \\
\leq \;& \sum_{i=1}^{|\mathbb{Q}|-1} \Big[T_{[\underline{Q},\overline{Q}]}(Q_{i+1}|\mathrm{X}) - T_{[\underline{Q},\overline{Q}]}(Q_i|\mathrm{X})\Big] \, (Q_{i+1} - Q_i)^2 \\
\leq \;& \sum_{i=1}^{|\mathbb{Q}|-1} \Big[T_{[\underline{Q},\overline{Q}]}(Q_{i+1}|\mathrm{X}) - T_{[\underline{Q},\overline{Q}]}(Q_i|\mathrm{X})\Big] \, (Q_{i+1} - Q_i)(\overline{Q} - \underline{Q}) \\
= \;& (\overline{Q} - \underline{Q}) \, \mathrm{ELTI}(\mathbb{Q})
\end{aligned}
$$

Decomposition of $TR(\hat{\theta})$ shows that shorter length of ELTI($\mathbb{Q}$) results in smaller upper bound of the total Bayes risk.

국문 초록

Holt and Laury(2002)의 논문에서 제시된 다중가격리스트(MPL)는 경제적 선호를 추론하는데 있어 널리 쓰이고 있는 방식이다. 하지만 이 방식은 Andersen et al.(2006)이 지적했던 바와 같이 구간 추정치만을 얻을 수 있고, framing effect에 취약하다. Harrison et al.(2005)이 구간 추정을 더 정확히 할 수 있는 반복다중가격리스트(iMPL)를 제시하긴 하였지만, iMPL은 이를 위해 MPL보다 더 많은 수의 경제적 선택을 피실험자에게 요구하며 피실험자의 업무 부담을 가중시킨다. 이 논문에서는 MPL과 같거나 더 적은 수의 문항만으로도 MPL보다 더 정확한 구간 추정을 가능하게 하는 새로운 방법을 제시한다. 새로운 방법이 이를 가능하게 해 주는 이유는 이미 알려져 있거나, 실험 중 새롭게 알게 되는 모든 정보를 활용하여 최적의 질문을 하기 때문이다. 이러한 과정은 베이즈 업데이트와 순차적인 질문 구조의 형태로 새로운 방법에 포함되어 있다.

주요어 : MPL, iMPL, 경제적 선호, 실험, 구간 추정, 순차적 질문 구조
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